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The Italian bondage and reinforcement numbers of digraphs

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Abstract

An *Italian dominating function* on a digraph D with vertex set $V(D)$ is defined as a function $f : V(D) \to \{0, 1, 2\}$ such that every vertex $v \in V(D)$ with $f(v) = 0$ has at least two in-neighbors assigned 1 under f or one in-neighbor w with $f(w) = 2$. The *weight* of an Italian dominating function f is the value $\omega(f) = f(V(D)) = \sum_{u \in V(D)} f(u)$. The *Italian domination number* of a digraph D, denoted by $\gamma_I(D)$, is the minimum taken over the weights of all Italian dominating functions on D. The *Italian bondage number* of a digraph D, denoted by $b_I(D)$, is the minimum number of arcs of $A(D)$ whose removal in D results in a digraph D' with $\gamma_I(D') > \gamma_I(D)$. The *Italian reinforcement number* of a digraph D, denoted by $r_I(D)$, is the minimum number of extra arcs whose addition to D results in a digraph D' with $\gamma_I(D') < \gamma_I(D)$. In this paper, we initiate the study of Italian bondage and reinforcement numbers in digraphs and present some bounds for $b_I(D)$ and $r_I(D)$. We also determine the Italian bondage and reinforcement numbers of some classes of digraphs.

Keywords: Italian domination number, Italian bondage number, Italian reinforcement number, domination number
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1. Introduction

Let $D = (V, A)$ be a finite simple digraph with vertex set $V = V(D)$ and arc set $A = A(D)$. The order $n(D)$ of a digraph is the size of $V(D)$. For an arc $uv \in A(D)$, we say that v is an *out-neighbor* of u and u is an *in-neighbor* of v. We denote the set of in-neighbors and out-neighbors

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of v by $N_D^-(v)$ and $N_D^+(v)$, respectively. We write $deg_D^-(v)$ and $deg_D^+(v)$ for the size of $N_D^-(v)$ and $N_D^+(v)$, respectively. Let $N_D^-[v] = N_D^-(v) \cup \{v\}$ and $N_D^+[v] = N_D^+(v) \cup \{v\}$. For s subset S of $V(D)$, we define $N^+(S) = \bigcup_{v \in S} N_D^+(v)$ and $N^+[S] = \bigcup_{v \in S} N_D^+[v]$. The *maximum out-degree* and *maximum in-degree* of a digraph D are denoted by $\Delta^+(D)$ and $\Delta^-(D)$, respectively.

For a digraph D, a subset S of $V(D)$ is a *dominating set* if $\bigcup_{v \in S} N_D^+[v] = V(D)$. The *domination number* $\gamma(D)$ is the minimum cardinality of a dominating set of D. The concept of the domination number of a digraph was introduced in [2]. The *bondage number* $b(D)$ of a digraph D is the minimum number of arcs of $A(D)$ whose removal in D results in a digraph D' with $\gamma(D') > \gamma(D)$. The concept of the bondage number of a digraph was proposed in [1]. The *reinforcement number* $r(D)$ of a digraph D is the minimum number of extra arcs whose addition to D results in a digraph D' with $\gamma(D') < \gamma(D)$. The concept of the reinforcement number of a digraph was introduced in [6].

Among the variations of domination, so called Italian domination of graphs is introduced in [3]. The authors of [3] present bounds relating the Italian domination number to some other domination parameters. The authors of [5] characterize the trees T for which $\gamma(T) + 1 = \gamma_I(T)$ and also characterize the trees T for which $\gamma_I(T) = 2\gamma(T)$. After that, there are many studies on Italian domination of graphs in [7, 8, 10, 14, 15]. Recently, the author of [16] initiated the study of the Italian domination number in digraphs. Related results was given in [9, 17, 12, 13]. Our aim in this paper is to initiate the study of Italian bondage and reinforcement numbers for digraphs.

An *Italian dominating function* (IDF) on a digraph D with vertex set $V(D)$ is defined as a function $f : V(D) \to \{0, 1, 2\}$ such that every vertex $v \in V(D)$ with $f(v) = 0$ has at least two in-neighbors assigned 1 under f or one in-neighbor w with $f(w) = 2$. An Italian dominating function $f : V(D) \to \{0, 1, 2\}$ gives an ordered partition (V_0, V_1, V_2) (or (V_0^f, V_1^f, V_2^f) to refer to f) of $V(D)$, where $V_i := \{x \in V(D) \mid f(x) = i\}$. The *weight* of an Italian dominating function f is the value $\omega(f) = f(V(D)) = \sum_{u \in V(D)} f(u)$. The *Italian domination number* of a digraph D, denoted by $\gamma_I(D)$, is the minimum taken over the weights of all Italian dominating functions on D. A $\gamma_I(D)$*-function* is an Italian dominating function on D with weight $\gamma_I(D)$.

For a graph G, the *associated digraph* G^* is the digraph obtained from G by replacing each edge of G by two oppositely oriented arcs. It is easy to see that $\gamma_I(G^*)$ is equal to the Italian domination number of G. We naturally extend concepts given [11, 4] to digraphs and provide the definition as follows.

The *Italian bondage number* of a digraph D, denoted by $b_I(D)$, is the minimum number of arcs of $A(D)$ whose removal in D results in a digraph D' with $\gamma_I(D') > \gamma_I(D)$.

The *Italian reinforcement number* of a digraph D, denoted by $r_I(D)$, is the minimum number of extra arcs whose addition to D results in a digraph D' with $\gamma_I(D') < \gamma_I(D)$. The Italian reinforcement number of a digraph D is defined to be 0 if $\gamma_I(D) \leq 2$. A subset R of $A(\overline{D})$ is called an *Italian reinforcement set* (IRS) of D if $\gamma_I(D + R) < \gamma_I(D)$. An $r_I(D)$-set is an IRS of D with size $r_I(D)$.

This paper is organized as follows. In Section 2, we prepare basic results on the Italian domination number. In Section 3, we give some bounds of the Italian bondage number and determine the exact values of Italian bondage numbers of some classes of digraphs. In Section 4, we characterize all digraphs D with $r_I(D) = 1$. We give some bounds of the Italian reinforcement number and also determine the exact values of Italian reinforcement numbers of compositions of digraphs.

2. The Italian domination numbers

In this paper, we make use of the following results.

Observation 1. *For a digraph D, $\gamma_I(D) \leq n - \Delta^+(D) + 1$.*

Proof. Let D be a digraph, and let v be a vertex with $deg_D^+(v) = \Delta^+(D)$. Define a function $f : V(D) \to \{0, 1, 2\}$ by $f(v) = 2$, $f(x) = 0$ if $x \in N^+(v)$, and $f(x) = 1$ otherwise. It is easy to see that f is an IDF of D. □

The following result is the exact value of Italian domination number of a complete bipartite graph (see [3] for the definition of Italian dominating function and domination number on a graph).

Lemma 2.1 ([4])**.** *For a complete bipartite graph $K_{m,n}$ with $1 \leq m \leq n$ and $n \geq 2$,*

$$\gamma_I(K_{m,n}) = \begin{cases} 2, & \text{if } m \leq 2, \\ 3, & \text{if } m = 3, \\ 4, & \text{if } m \geq 4. \end{cases}$$

Theorem 2.1 ([16])**.** *Let D be a digraph of order n. Then $\gamma_I(D) \geq \lceil \frac{2n}{2+\Delta^+(D)} \rceil$.*

Theorem 2.2. *Let D be a digraph of order $n \geq 3$. Then $\gamma_I(D) = 2$ if and only if $\Delta^+(D) = n - 1$ or there exist two distinct vertices u and v such that $V(D) \setminus \{u, v\} \subseteq N_D^+(u)$ and $V(D) \setminus \{u, v\} \subseteq N_D^+(v)$.*

Proof. If $\Delta^+(D) = n - 1$ or there exist two distinct vertices u and v such that $V(D) \setminus \{u, v\} \subseteq N_D^+(u)$ and $V(D) \setminus \{u, v\} \subseteq N_D^+(v)$, then it is easy to see that $\gamma_I(D) = 2$.

Assume that $\gamma_I(D) = 2$. Let (V_0, V_1, V_2) be a $\gamma_I(D)$-function. Then $\gamma_I(D) = 2 = |V_1| + 2|V_2|$ and $|V_2| \leq 1$. If $|V_2| = 1$, then $|V_1| = 0$ and hence $\Delta^+(D) = n - 1$. If $|V_2| = 0$, then $|V_1| = 2$ and, by the definition of IDF, there exist two distinct vertices u and v such that $V(D) \setminus \{u, v\} \subseteq N_D^+(u)$ and $V(D) \setminus \{u, v\} \subseteq N_D^+(v)$. □

Theorem 2.3 ([16])**.** *Let D be a digraph of order $n \geq 3$. Then $\gamma_I(D) < n$ if and only if $\Delta^+(D) \geq 2$ or $\Delta^-(D) \geq 2$.*

Corollary 2.1. *If D is a directed path or cycle of order n, then $\gamma_I(D) = n$.*

3. The Italian bondage numbers

3.1. Bounds of the Italian bondage numbers

The *underlying graph* $G[D]$ of a digraph D is the graph obtained by replacing each arc uv by an edge uv. Note that $G[D]$ has two parallel edges uv when D contains the arc uv and vu. A digraph D is *connected* if the underlying graph $G[D]$ is connected. For a graph G, we denote the degree of $v \in V(G)$ by $deg_G(v)$. In particular, $\Delta(G)$ means the maximum degree in G.

Theorem 3.1. *If D is a digraph, and xyz a path of length 2 in $G[D]$ such that $yx, yz \in A(D)$, then*

$$b_I(D) \leq deg_{G[D]}(x) + deg_D^-(y) + deg_{G[D]}(z) - |N^-(x) \cap N^-(y) \cap N^-(z)|.$$

Moreover, if x and z are adjacent in $G[D]$, then

$$b_I(D) \leq deg_{G[D]}(x) + deg_D^-(y) + deg_{G[D]}(z) - 1 - |N^-(x) \cap N^-(y) \cap N^-(z)|.$$

Proof. Let B be the set of all arcs incident with x or z and all arcs terminating at y with the exception of all arcs from $N^-(x) \cap N^-(z)$ to y. Then

$$|B| \leq deg_{G[D]}(x) + deg_D^-(y) + deg_{G[D]}(z) - |N^-(x) \cap N^-(y) \cap N^-(z)|$$

and

$$|B| \leq deg_{G[D]}(x) + deg_D^-(y) + deg_{G[D]}(z) - 1 - |N^-(x) \cap N^-(y) \cap N^-(z)|$$

when x and z are adjacent.

Let $D' = D - B$. In D', x and z are isolated, and all in-neighbors of y in D', if any, lie in $N^-(x) \cap N^-(z)$. Let $f = (V_0, V_1, V_2)$ be a $\gamma_I(D')$-function. Then $f(x) = f(z) = 1$. If $f(y) = 2$, then

$$(V_0 \cup \{x, z\}, V_1 \setminus \{x, z\}, V_2)$$

is an IDF of D with weight less than $\omega(f)$. If $f(y) = 1$, then

$$(V_0 \cup \{x, z\}, V_1 \setminus \{x, y, z\}, V_2 \cup \{y\})$$

is an IDF of D with weight less than $\omega(f)$. However, if $f(y) = 0$, then there exists $w \in N^-(x) \cap N^-(y) \cap N^-(z)$ such that $f(w) = 2$ or there exist $w_1, w_2 \in N^-(x) \cap N^-(y) \cap N^-(z)$ such that $f(w_1) = f(w_2) = 1$. Since w, w_1 and w_2 are in-neighbors of x and z in D,

$$(V_0 \cup \{x, z\}, V_1 \setminus \{x, z\}, V_2)$$

is an IDF of D with weight less than $\omega(f)$. This completes the proof. $\square$

Theorem 3.2. *Let D be a digraph of order $n \geq 3$. If $G[D]$ is connected, then*

$$b_I(D) \leq (\gamma_I(D) - 1)\Delta(G[D]).$$

Proof. We proceed by induction on $\gamma_I(D)$. Assume that $\gamma_I(D) = 2$. For a vertex $u \in V_1 \cup V_2$, let B_u be the set of arcs incident with u. Since $\gamma_I(D - u) \geq 2$ by $n \geq 3$, we have

$$\gamma_I(D - B_u) = \gamma_I(D - u) + 1 \geq 3.$$

This implies that $b_I(D) \leq |B_u|$ for $u \in V_1 \cup V_2$. Thus, $b_I(D) \leq \Delta(G[D])$.

Assume that the result is true for every digraph with the Italian domination number $k \geq 3$. Let D be a digraph with $\gamma_I(D) = k + 1$. Suppose to the contrary that $b_I(D) > (\gamma_I(D) - 1)\Delta(G[D])$. Let u be an arbitrary vertex of D, and let B_u be the set of arcs incident with u. Then we have $\gamma_I(D) = \gamma_I(D - B_u)$. Let f be a $\gamma_I(D - B_u)$-function. Then $f(u) = 1$ and the function f

restricted to $D-u$ is also a $\gamma_I(D-u)$-function. This implies that $\gamma_I(D-u)=\gamma_I(D)-1$. So, $b_I(D)\leq b_I(D-u)+deg_{G[D]}(u)$. By the induction hypothesis, we have

$$\begin{aligned} b_I(D) &\leq b_I(D-u)+deg_{G[D]}(u) \\ &\leq (\gamma_I(D-u)-1)\Delta(G[D-u])+deg_{G[D]}(u) \\ &\leq (\gamma_I(D-u)-1)\Delta(G[D])+\Delta(G[D]) \\ &= \gamma_I(D-u)\Delta(G[D]) \\ &= (\gamma_I(D)-1)\Delta(G[D]). \end{aligned}$$

This is a contradiction. □

3.2. The Italian bondage numbers of some classes of digraphs

For a graph G, the *associated digraph* G^* is the digraph obtained from G by replacing each edge of G by two oppositely oriented arcs. Note that $\gamma_I(G)=\gamma_I(G^*)$ for any graph G.

Theorem 3.3. *Let K_n^* be the complete digraph of order $n\geq 3$. Then $b_I(K_n^*)=n$.*

Proof. Note that $\gamma_I(K_n^*)=2$. Let B be an arc set of K_n^*. Define $D:=K_n^*-B$. If D contain a vertex x such that $deg_D^+(x)=n-1$, then it follows from Observation 1 that $\gamma_I(D)=2$. This implies that $b_I(K_n^*)\geq n$.

Let $\{x_1,x_1,\ldots,x_n\}$ be the vertex set of K_n^*, and let $B:=\{x_1x_2,x_2x_3,\ldots,x_nx_1\}$ be the arc set of a directed cycle in K_n^*. Define $D:=K_n^*-B$. Then one can observe that there do not exist two distinct vertices u and v in D such that $V(D)\setminus\{u,v\}\subseteq N_D^+(u)$ and $V(D)\setminus\{u,v\}\subseteq N_D^+(v)$. It follows from Theorem 2.2 that $\gamma_I(D)\geq 3$. This completes the proof. □

The following result follows from the definition of associated digraph and Lemma 2.1. For a complete bipartite digraph $K_{m,n}^*$ with $1\leq m\leq n$,

$$\gamma_I(K_{m,n}^*)=\begin{cases} 2, & \text{if } m\leq 2,\\ 3, & \text{if } m=3,\\ 4, & \text{if m}\geq 4.\end{cases} \tag{1}$$

Theorem 3.4. *Let $K_{m,n}^*$ be the complete bipartite digraph such that $1\leq m<n$. Then*

$$b_I(K_{m,n}^*)=\begin{cases} 1, & \text{if } m\leq 2,\\ 2, & \text{if } m=3,\\ m+2, & \text{if } m\geq 4.\end{cases}$$

Proof. We denote $K_{m,n}^*$ by D. Let $X=\{x_1,x_2,\ldots,x_m\}$ and $Y=\{y_1,y_2,\ldots,y_n\}$ be the partite sets of D. The result is clear for $m\leq 2$.

Assume that $m=3$. It follows from (1) that $\gamma_I(D)=3$. If we remove two arcs terminating at some vertex $y_j\in Y$, then the Italian domination number of resulting digraph increases. So, $b_I(D)\leq 2$. For any arc e of $A(D)$, there exist two vertices x_i and x_j such that $N_{D-e}^+(x_i)=n$ and $N_{D-e}^+(x_j)=n$. Thus, we have $b_I(D)=2$.

Assume that $m \geq 4$. It follows from (1) that $\gamma_I(D) = 4$. Let $B = \{x_i y_1 \mid 1 \leq i \leq m\} \cup \{y_1 x_1, y_1 x_2\}$. It is easy to see that $\gamma_I(D - B) \geq 5$. So, $b_I(D) \leq m + 2$.

Next, we show that $b_I(D) \geq m + 2$. Let B' be a subset of $A(D)$ such that $|B'| = m + 1$, and let $D' = D - B'$. Then D' has at least $n - 1$ vertices whose outdegree are equal in D and D'. Let $E = \{v \in V(D) \mid d^+_D(v) = d^+_{D'}(v)\}$. If $E \cap X \neq \emptyset \neq E \cap Y$, then clearly $\gamma_I(D') = 4$. Henceforth, we assume that $E \cap X = \emptyset$ or $E \cap Y = \emptyset$. Without loss of generality, assume that $E \cap X = \emptyset$. Then $E \subseteq Y$ and B' contains one outgoing arc for each $x_i \in X$. Since $|B'| = m + 1 < 2m$, B' contains exactly one outgoing arc for some $x_i \in X$. Without loss of generality, assume that $i = 1$ and $x_1 y_1 \in B'$. If $E = Y$, then

$$(V(D') \setminus \{x_1, y_1\}, \emptyset, \{x_1, y_1\})$$

is an IDF of D' with weight 4. Let $E \subset Y$. We may assume that $E \subseteq \{y_1, y_2, \ldots, y_{n-1}\}$. Thus, B' contains one outgoing arc from y_n, say $y_n x_m$. Since $|B'| = m + 1$, B' contains exactly one outgoing arc for each $x_i \in X$ and one outgoing arc from y_n. If $x_i y_j \in B'$ for some $1 \leq i \leq m$ and $j < n$, then

$$(V(D') \setminus \{x_i, y_j\}, \emptyset, \{x_i, y_j\})$$

is an IDF of D' with weight 4. Thus, we assume that $x_i y_n \in B'$ for each $1 \leq i \leq m$. But,

$$(V(D') \setminus \{x_m, y_n\}, \emptyset, \{x_m, y_n\})$$

is an IDF of D' with weight 4. Thus, we have $b_I(D) \geq m + 2$. □

4. The Italian reinforcement numbers

4.1. Digraphs with $r_I(D) = 1$

Lemma 4.1. *Let D be a digraph with $\gamma_I(D) \geq 3$. Let F be an $r_I(D)$-set, and let g be a $\gamma_I(D)$-function of $D + F$. Then the following hold:*

1. *For each arc $v_1 v_2 \in F$, $g(v_1) \neq 0$ and $g(v_2) = 0$.*
2. $\gamma_I(D + F) = \gamma_I(D) - 1$.

Proof. If there exists an arc $v_1 v_2 \in F$ such that either $g(v_i) \geq 1$ for each $i \in \{1, 2\}$ or $g(v_1) = g(v_2) = 0$, then g is also an IDF of $D + (F \setminus \{v_1 v_2\})$, and hence $F \setminus \{v_1 v_2\}$ is an IRS of D, which contradicts the definition of F. Thus, (i) holds.

By the definition of F, we have $\gamma_I(D+F) \leq \gamma_I(D) - 1$. Suppose that $\gamma_I(D+F) \leq \gamma_I(D) - 2$. Let $v_1 v_2 \in F$. By (i), $g(v_1) \neq 0$ and $g(v_2) = 0$. Then the function $g' : V(D + (F \setminus \{v_1 v_2\})) \to \{0, 1, 2\}$ with

$$g'(x) = \begin{cases} 1, & \text{if } x = v_2, \\ g(x), & \text{otherwise.} \end{cases}$$

is an IDF of $D + (F \setminus \{v_1 v_2\})$ such that $\omega(g') = \omega(g) + 1 \leq \gamma_I(D) - 1$. This implies that $F \setminus \{v_1 v_2\}$ is an IRS of D, which contradicts the definition of F. Thus, (ii) holds. □

Lemma 4.2. *Let D be a digraph of order $n \geq 3$, $\Delta^+(D) \geq 1$ and $\gamma_I(D) = n$. Then $r_I(D) = 1$.*

Proof. It follows from Theorem 2.3 that $\Delta^+(D) = 1$. Since $\sum_{v\in V(D)} deg^+(v) = \sum_{v\in V(D)} deg^-(v)$, we have $\Delta^-(D) \geq 1$. It also follows from Theorem 2.3 that $\Delta^-(D) = 1$. Thus, D is disjoint union of directed paths, cycles or isolated vertices. Let $uv \in A(D)$ and $w \in V(D) \setminus \{u, v\}$. It is easy to see that

$$(\{v, w\}, V(D) \setminus \{u, v, w\}, \{u\})$$

is an IDF of $D + uw$ with weight $n - 1$. Thus, we have $r_I(D) = 1$. □

Theorem 4.1. *Let D be a digraph with $\gamma_I(D) \geq 3$. Then $r_I(D) = 1$ if and only if there exist a $\gamma_I(D)$-function $f = (V_0, V_1, V_2)$ of D and a vertex $v \in V_1$ satisfying one of the following conditions:*

1. $f(N^-(v)) = 1$ *and* $f(N^-(x) \setminus \{v\}) \geq 2$ *for each* $x \in N^+(v) \cap V_0$.
2. $f(N^-(v)) = 0$*,* $f(N^-(x) \setminus \{v\}) \geq 2$ *for each* $x \in N^+(v)$*, and* $V_2 \neq \emptyset$.

Proof. First, assume that (i) holds. Then it follows from $f(N^-(v)) = 1$ that there exists $u \in V_1 \cap N^-(v)$. Since $\gamma_I(D) \geq 3$, there exists $w \in (V_1 \cup V_2) \setminus \{v, u\}$. Since $uv \in A(D)$ and $f(N^-(x) \setminus \{v\}) \geq 2$ for each $x \in N^+(v) \cap V_0$,

$$(V_0 \cup \{v\}, V_1 \setminus \{v\}, V_2)$$

is an IDF of $D + wv$ with weight $\gamma_I(D) - 1$. Thus, we have $r_I(D) = 1$.

Next, assume that (ii) holds. Let $w \in V_2$. Then it follows from $f(N^-(v)) = 0$ that $wv \notin A(D)$. Since $f(N^-(x) \setminus \{v\}) \geq 2$ for each $x \in N^+(v)$,

$$(V_0 \cup \{v\}, V_1 \setminus \{v\}, V_2)$$

is an IDF of $D + wv$ with weight $\gamma_I(D) - 1$. Thus, we have $r_I(D) = 1$.

Conversely, assume that $r_I(D) = 1$, and let uv be an arc of D with $\gamma_I(D + uv) < \gamma_I(D)$. Let g be a $\gamma_I(D + uv)$-function. Then $g(u) \neq 0$ and $g(v) = 0$ by Lemma 4.1(i). The function $f : V(D) \to \{0, 1, 2\}$ with

$$f(x) = \begin{cases} 1, & \text{if } x = v, \\ g(x), & \text{otherwise.} \end{cases}$$

is an IDF of D. It follows from Lemma 4.1(ii) that f is a $\gamma_I(D)$-function.

Suppose that $f(N^-(v)) \geq 2$. Then $g(N^-(v)) \geq 2$. So, g is an IDF of D. This means that $\gamma_I(D) \leq \omega(g) = \gamma_I(D + uv)$, a contradiction. Thus, we have $f(N^-(v)) \leq 1$.

Note that $f(N^-(x) \setminus \{v\}) = g(N^-(x) \setminus \{v\}) \geq 2$ for each $x \in N^+(v) \cap V_0^f$, since g is a $\gamma_I(D + uv)$-function with $g(v) = 0$. If $f(N^-(v)) = 1$, then (i) holds. Now assume that $f(N^-(v)) = 0$. Then we have $g(u) = f(u) = 2$, since $g(v) = 0$ and u is an in-neighbor of v in $D + uv$. As $V_2^f \neq \emptyset$, (ii) holds. □

4.2. Bounds of the Italian reinforcement numbers

Theorem 4.2. *If D is a digraph of order n with $\gamma_I(D) \geq 3$, then*

$$r_I(D) \leq n - \Delta^+(D) - \gamma_I(D) + 2.$$

Proof. Since $\gamma_I(D) \geq 3$, it follows from Theorem 2.2 that $\Delta^+(D) \leq n-2$. Let u be a vertex with $deg^+_D(u) = \Delta^+(D)$ and let $R = \{uv \mid v \in V(D) \setminus N^+[u]\}$. Then $(V(D) \setminus \{u\}, \emptyset, \{u\})$ is an IDF of $D + R$. Thus,

$$r_I(D) \leq n - \Delta^+(D) - 1.$$

There exist $r_I(D) - 1$ vertices $v_1, v_2, \ldots, v_{r_I(D)-1}$ in $V(D) \setminus N^+[u]$.

Let D' be a digraph obtained from D by adding $r_I(D) - 1$ arcs uv_i. Then, by the definition of $r_I(D)$ and Observation 1,

$$\gamma_I(D) = \gamma_I(D') \leq n - \Delta^+(D') + 1.$$

Since $\Delta^+(D') = \Delta^+(D) + r_I(D) - 1$, we have $r_I(D) \leq n - \Delta^+(D) - \gamma_I(D) + 2$. □

Theorem 4.3. *If D is a digraph such that $\gamma_I(D) = 3$ and $\gamma(D) = 2$, then $r(D) \leq r_I(D) + 1$.*

Proof. Let R be a $r_I(D)$-set. Then $\gamma_I(D + R) = 2$. If $r \in R$, then clearly $r_I(D + (R \setminus \{r\})) = 1$.

By Theorem 4.1, there exist a $\gamma_I(D + (R \setminus \{r\}))$-function $f = (V_0, V_1, V_2)$ of $D + (R \setminus \{r\})$ and a vertex $v \in V_1$ satisfying one of the following conditions:

1. $f(N^-(v)) = 1$ and $f(N^-(x) \setminus \{v\}) \geq 2$ for each $x \in N^+(v) \cap V_0$.
2. $f(N^-(v)) = 0$, $f(N^-(x) \setminus \{v\}) \geq 2$ for each $x \in N^+(v)$, and $V_2 \neq \emptyset$.

Suppose that (i) holds. Since $\gamma_I(D + (R \setminus \{r\})) = 3$, it follows from $f(N^-(v)) = 1$ that there exists $u \in V_1$ such that $u \notin N^-(v)$. Let $w \in V_1 \cap N^-(v)$. Since $f(N^-(x) \setminus \{v\}) \geq 2$ for each $x \in N^+(v) \cap V_0$, we have $ux, wx \in A(D + (R \setminus \{r\})$ for each $x \in N^+(v) \cap V_0$. Since $\gamma_I(D + (R \setminus \{r\})) = 3$, we have $u, w \in N^-(x)$ for each $x \in V_0 \setminus N^+(v)$. Thus, $\{u\}$ is a dominating set of $D + ((R \setminus \{r\}) \cup \{uv, uw\})$. This implies that $r(D) \leq r_I(D) + 1$.

Suppose that (ii) holds. Let $V_2 = \{u\}$. Then we have $V_0 \subseteq N^+(u)$. Thus, $\{u\}$ is a dominating set of $D + ((R \setminus \{r\}) \cup \{uv\})$. This implies that $r(D) \leq r_I(D)$. □

4.3. The Italian reinforcement numbers of compositions of digraphs

For two digraphs G and H, two kinds of joins $G \rightarrow H$ and $G \leftrightarrow H$ were defined in [6]. The digraph $G \rightarrow H$ consists of G and H with extra arcs from each vertex of G to every vertex of H. The digraph $G \leftrightarrow H$ can be obtained from $G \rightarrow H$ by adding arcs from each vertex of H to every vertex of G.

Theorem 4.4. *Let G and H be two digraphs such that $\Delta^+(G) \geq 1$ and $\Delta^+(H) \geq 1$. Then*

1. $\gamma_I(G \rightarrow H) = \gamma_I(G)$,
2. $r_I(G \rightarrow H) = r_I(G)$,

Proof. (i) Let f be a $\gamma_I(G)$-function. Then it follows from the definition of IDF that f is extended to an IDF of $G \rightarrow H$ by assigning 0 to every vertex of H. Thus, $\gamma_I(G \rightarrow H) \leq \gamma_I(G)$. On the other hand, if $g = (V_0, V_1, V_2)$ is a $\gamma_I(G \rightarrow H)$-function, then clearly $g|_G := (V_0 \cap V(G), V_1 \cap V(G), V_2 \cap V(G))$ is an IDF of G. Thus, $\gamma_I(G) \leq \gamma_I(G \rightarrow H)$.

(ii) If $\gamma_I(G)=2$, then it follows from (i) that $\gamma_I(G\to H)=2$. So, $r_I(G\to H)=r_I(G)$. From now on, we assume $\gamma_I(G)\geq 3$. Let R be a $r_I(G)$-set. Then

$$\gamma_I((G\to H)+R)=\gamma_I((G+R)\to H)=\gamma_I(G+R)<\gamma_I(G)=\gamma_I(G\to H).$$

Thus, $r_I(G\to H)\leq r_I(G)$.

Now we claim that $r_I(G)\leq r_I(G\to H)$. Let R_1 be a $r_I(G\to H)$-set. Suppose that R_2 is a subset of R_1 such that two ends of arcs in R_2 lie in $V(G)$. Let $f=(V_0^f,V_1^f,V_2^f)$ be a $\gamma_I((G\to H)+R_1)$-function, and let $g=f|_G$. We divide our consideration into the following two cases.

Case 1. g is an IDF of $G+R_2$.

Then we have

$$\begin{aligned}\gamma_I((G\to H)+R_1) &= \omega(f)\\ &\geq \omega(g)\\ &\geq \gamma_I(G+R_2)\\ &= \gamma_I((G+R_2)\to H)\\ &= \gamma_I((G\to H)+R_2)\\ &\geq \gamma_I((G\to H)+R_1).\end{aligned}$$

Since $R_2\subseteq R_1$ and R_1 is a $r_I(G\to H)$-set, we have $R_1=R_2$. So, $\gamma_I(G+R_2)\leq\omega(g)=\gamma_I((G\to H)+R_2)<\gamma_I(G\to H)=\gamma_I(G)$. Thus, $r_I(G)\leq|R_2|=|R_1|=r_I(G\to H)$.

Case 2. g is not an IDF of $G+R_2$.

Then some vertex $u\in V_0^f\cap V(G)$ has an in-neighbor $w\in V(H)$ such that $wu\in R_1$. Fix $v\in V(G)$, and let $R_3=\{vu\mid u\in N\}$, where $N=\{u\in V_0^f\cap V(G)\mid u$ does not dominated by the vertices of G under $f\}$. Then clearly $|R_2\cup R_3|\leq|R_1|$. It is easy to see that the function $h:V(G)\to\{0,1,2\}$ defined by $h(v)=max\{f(v),max\{f(N^-(u)\cap V(H))\mid u\in N\}\}$ and $h(x)=f(x)$ otherwise, is an IDF of $G+(R_2\cup R_3)$ with weight at most $\omega(f)$. Now we have

$$\begin{aligned}\gamma_I(G+(R_2\cup R_3)) &\leq \omega(h)\\ &\leq \omega(f)\\ &= \gamma_I((G\to H)+R_1)\\ &< \gamma_I(G\to H)\\ &= \gamma_I(G).\end{aligned}$$

Thus, $r_I(G)\leq|R_2\cup R_3|\leq|R_1|=r_I(G\to H)$. □

The corona $G\overrightarrow{\circ}H$ of two digraphs G and H is formed from one copy of G and $n(G)$ copies of H by joining v_i to every vertex of H_i, where v_i is the ith vertex of G and H_i is the ith copy of H.

Theorem 4.5. *Let G and H be two digraphs with $n(H)\geq 2$. Then*

1. $\gamma_I(G\overrightarrow{\circ}H)=2n(G)$,

2.
$$r_I(G\overrightarrow{\circ}H)=\begin{cases}0, & \textit{if } n(G)=1,\\ n(H,) & \textit{if } G \textit{ is the empty digraph and } n(G)\geq 2,\\ n(H)-1, & \textit{otherwise.}\end{cases}$$

Proof. (i) If $n(G)=1$, then clearly $\gamma_I(G\overrightarrow{\circ}H)=2$. Assume that $n(G)\geq 2$. It is easy to see that $(V(G\overrightarrow{\circ}H)\setminus V(G),\emptyset,V(G))$ is an IDF of $G\overrightarrow{\circ}H$. So, $\gamma_I(G\overrightarrow{\circ}H)\leq 2n(G)$.

Let f be a $\gamma_I(G\overrightarrow{\circ}H)$-function. To dominate the vertices of H_i,we must have $\sum_{x\in V(H_i)\cup\{v_i\}} f(x)\geq 2$. Since a single vertex of G does not dominate vertices in different copies of H, we have $\gamma_I(G\overrightarrow{\circ}H)\geq 2n(G)$.

(ii) If $n(G)=1$, then clearly $r_I(G\overrightarrow{\circ}H)=0$. Assume that $n(G)\geq 2$. We divide our consideration into the following two cases.

Case 1. $A(G)=\emptyset$.

Let $R=\{v_1u\mid u\in V(H_{n(G)})\}$. Then it is easy to see that
$$(\bigcup_{i=1}^{n(G)} V(H_i),\{v_{n(G)}\},V(G)\setminus\{v_{n(G)}\})$$
is an IDF of $(G\overrightarrow{\circ}H)+R$ with weight $2n(G)-1$. Thus, $r_I(G\overrightarrow{\circ}H)\leq n(H)$.

Let F be a $r_I(G\overrightarrow{\circ}H)$-set. By Lemma 4.1(ii), $\gamma_I((G\overrightarrow{\circ}H)+F)=\gamma_I(G\overrightarrow{\circ}H)-1$. Let $U_i=\{v_i\}\cup V(H_i)$ for $1\leq i\leq n(G)$, and let $f=(V_0,V_1,V_2)$ be a $r_I((G\overrightarrow{\circ}H)+F)$-function. Then $\sum_{x\in U_i} f(x)\leq 1$ for some i, say $i=n(G)$. To dominate the vertices in $U_{n(G)}$, F must contain at least $n(H)$ arcs which go from some vertices in $(V_1\cup V_2)\cap(\bigcup_{i=1}^{n(G)-1}U_i)$ to vertices in $U_{n(G)}$. Thus, $|F|\geq n(H)$ and so $r_I(G\overrightarrow{\circ}H)\geq n(H)$.

Case 2. $A(G)\neq\emptyset$.

Without loss of generality, we assume that $v_1v_{n(G)}\in A(G)$. Let $V(H_{n(G)})=\{w_1,\ldots,w_{n(H)}\}$, and let $R=\{v_1w_j\mid w_j\in V(H_{n(G)})\setminus\{w_1\}\}$. Then
$$(\bigcup_{i=1}^{n(G)} V(H_i)\cup\{v_{n(G)}\},\{w_1\},\{v_1,\ldots,v_{n(G)-1}\})$$
is an IDF of $(G\overrightarrow{\circ}H)+R$ with weight $2n(G)-1$. Thus, $r_I(G\overrightarrow{\circ}H)\leq n(H)-1$. By using the same argument given in Case 1, one can show that $r_I(G\overrightarrow{\circ}H)\geq n(H)-1$. □

5. Conclusion

Italian domination in digraphs has been less explored if compared to its counterpart in undirected grphs. In this paper, we define the Italian bondage number and the Italian reinforcement number in digraphs. We provide general bounds of the Italian bondage number and determine exact values of the Italian bondage number for specific classes of digraphs. We also characterize cases where the Italian reinforcement number is one, and establish upper bounds in terms of digraph parameters. In the direction of further research, we propose a study on the Italian domination number of various digraph products. We also naturally propose a study on the Italian bondage number and the Italian reinforcement number in the products of digraphs.

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