



A bound on connectivity of iterated line graphs

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Abstract

For simple connected graphs that are neither paths nor cycles, we define $l(G) = \max\{m : G \text{ has a divalent path of length } m \text{ that is not both of length } 2 \text{ and in a } K_3\}$, where a divalent path is a path whose internal vertices have degree two in G . Let G be a graph and $L^n(G)$ be its n -th iterated line graph of G . We use $\kappa'_e(G)$ and $\kappa(G)$ for the essential edge connectivity and vertex connectivity of G , respectively. Let G be a simple connected graph that is not a path, a cycle or $K_{1,3}$, with $l(G) = l \geq 1$. We prove that (i) for integers $s \geq 1$, $\kappa'_e(L^{l+s}(G)) \geq 2^s + 2$; (ii) for integers $s \geq 2$, $\kappa(L^{l+s}(G)) \geq 2^{s-1} + 2$. The bounds are best possible.

Keywords: iterated line graphs, essential edge connectivity, vertex connectivity

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1. Introduction

We use [1] for terminology and notation not defined here, and consider finite and simple graphs only. Let A, B be two vertex sets in a graph G , and $[A, B]$ denote the set of edges with one end in A and the other in B . In particular, $\kappa(G)$ and $\kappa'(G)$ represent the *connectivity* and *edge-connectivity* of a graph G , respectively. A graph is *trivial* if it contains no edges. An edge cut Y of G is *essential* if $G - Y$ has at least two nontrivial components. For an integer $k > 0$, a graph G is *essentially k -edge-connected* if G does not have an essential edge cut Y with $|Y| < k$. We use $\kappa'_e(G)$ to

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denote the *essential edge connectivity* of a graph G . A degree sum property follows immediately from the definition of essential edge connectivity.

Proposition 1.1. (Shao, Proposition 2.1 of [4]) *Let $n \geq 1$ be an integer and G be a graph which is not $K_{1,n-1}$ or K_3 . Then the degree sum of any two adjacent vertices is at least $\kappa'_e(G) + 2$.*

The *line graph* of a graph G , denoted by $L(G)$ or $L^1(G)$, has $E(G)$ as its vertex set, where two vertices in $L(G)$ are adjacent if and only if the corresponding two edges in G have a common vertex. Iteratively, $L^n(G) = L(L^{n-1}(G))$ and $L^0(G) = G$ for integers $n \geq 1$. The following connectivity properties can be derived from the definitions.

Proposition 1.2. (Shao, Proposition 1.2 of [4]) *Let $n \geq 1$ be an integer and G be a graph which is not K_3 or $K_{1,n-1}$. Then each of the following holds.*

- (i) $\kappa'_e(G) \geq \kappa'(G)$.
- (ii) $\kappa'_e(G) = \kappa(L(G))$.
- (iii) $\kappa'_e(L(G)) \geq \kappa'_e(G)$.
- (iv) $\kappa(L(G)) \geq \kappa(G)$.

Proposition 1.2(ii) indicates that it is useful to investigate properties of essential edge connectivity in order to obtain properties of vertex connectivity in line graphs. Here is a result on how essential edge connectivity grows for iterated line graphs.

Theorem 1.1. (Shao, Theorem 1.3 of [4]) *Let $n \geq 1$ be an integer and G be a graph which is not $K_{1,n-1}$ or K_3 . If G does not contain degree two vertices, then $\kappa'_e(L(G)) \geq 2\kappa'_e(G) - 2$.*

For simple connected graphs that are neither paths nor cycles, we define $l(G) = \max\{m : G \text{ has a divalent path of length } m \text{ that is not both of length } 2 \text{ and in a } K_3\}$, where a divalent path is a path whose internal vertices have degree two in G . We call $l(G)$ the *divalent length* of G .

For the graph G depicted in Figure 1, $\kappa'_e(L(G)) = \kappa'_e(G) = 3$; the one in Figure 1, $\kappa'_e(L(G)) = \kappa'_e(G) = 1$. In many cases as these graphs depicted in Figures 0 and 1, when G has a divalent path with internal vertices of degree two, the connectivity of the line graphs stays the same. Since each iteration will reduce the lengths of all divalent paths by one, after $l(G) - 1$ times of iterations, the internal vertices of degree 2 either disappear or lie in a triangle, and the connectivity is likely to increase.

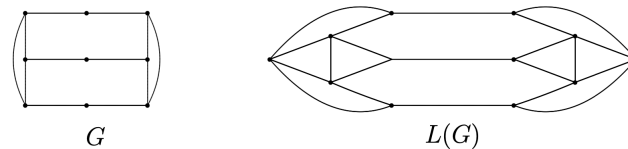


Figure 1. An example for $\kappa'_e(L(G)) = \kappa'_e(G)$.

A former result gives a bound on the essential edge connectivity of the $(l + 1)$ -th iterated line graphs.

Proposition 1.3. (Zhang et al., Lemma 2.5 of [6]) Let G be a connected graph with $l(G) = l$. Then $\kappa'_e(L^{l+1}(G)) \geq 4$.

In this note, we obtain the bounds for essential edge connectivity and connectivity of iterated line graphs.

Theorem 1.2. Let G be a simple connected graph that is not a path, a cycle or $K_{1,3}$, with $l(G) = l \geq 1$. Then each of the following holds:

- (i) For integers $s \geq 1$, $\kappa'_e(L^{l+s}(G)) \geq 2^s + 2$. The bound is best possible.
- (ii) For integers $s \geq 2$, $\kappa(L^{l+s}(G)) \geq 2^{s-1} + 2$. The bound is best possible.

Note that (i) does not hold for $s = 0$ and (ii) does not hold for $s = 1$ as there exist graphs with $\kappa'_e(L^l(G)) = \kappa(L^{l+1}(G)) = 2$. As shown in Figure 2(a), $l(G) = 2$ and in Figure 2(c), $\kappa'_e(L^2(G)) = 2$. By Proposition 1.2(ii), $\kappa(L^3(G)) = \kappa'_e(L^2(G)) = 2$.

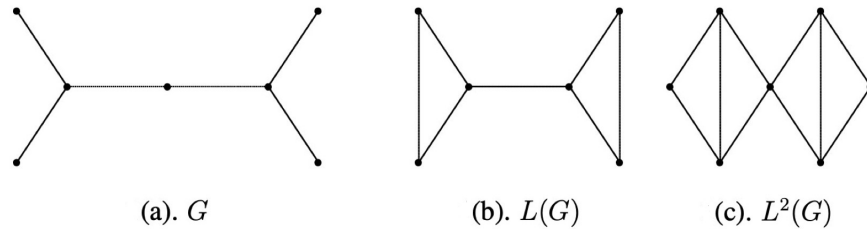


Figure 2. An example to show (i) does not hold for $s = 0$.

2. Previous Results

For a graph G and a vertex $v \in V(G)$, define

$$N_G(v) = \{u \in V(G) : u \text{ is adjacent to } v \text{ in } G\}$$

and

$$E_G(v) = \{e \in E(G) : e \text{ is incident with } v \text{ in } G\}.$$

Let $L(G)$ be the line graph of G , and

$$X \text{ be a minimal essential edge cut of } L(G). \tag{1}$$

By (1), $L(G) - X$ has exactly two nontrivial components, so we assume

$$L_1, L_2 \text{ are the only two nontrivial components of } L(G) - X. \tag{2}$$

By (2), $V(L(G))$ is a disjoint union of $V(L_1)$ and $V(L_2)$. By the definition of line graphs, $E(G)$ is a disjoint union of two edge sets corresponding to $V(L_1)$ and $V(L_2)$. For each edge $e \in E(G)$, let $v_e \in V(L(G))$ denote its corresponding vertex in the line graph $L(G)$. Let $f : E(G) \mapsto \{1, 2\}$ be a 2-edge-coloring of G such that

$$f(e) = i \text{ if and only if } v_e \in V(L_i) \text{ for } i = 1, 2.$$

Proposition 2.1. (Shao, [3]) Let G be a graph, $e_1, e_2 \in E(G)$ and $v_{e_1}, v_{e_2} \in V(L(G))$ be their corresponding vertices in the line graph $L(G)$. Then each of the following holds:

- (i) For $i = 1, 2$, $|\{e \in E(G) : f(e) = i\}| = |V(L_i)| \geq 2$.
- (ii) $v_{e_1}v_{e_2} \in X$ if and only if e_1, e_2 share a common vertex and $f(e_1) \neq f(e_2)$.

A vertex of a graph G is *mono-colored* if all incident edges have the same color in G , and a vertex is *bi-colored* if the incident edges have two different colors.

Proposition 2.2. (Shao, [3]) Let G be a graph, V_{12} the set of all bi-colored vertices of G , or equivalently,

$$V_{12} = \{v \in V(G) : \text{there exist } e_1, e_2 \in E_G(v) \text{ such that } f(e_1) = 1 \text{ and } f(e_2) = 2\}. \quad (3)$$

Then each of the following holds.

- (i) $d_G(v) \geq 2$ for each $v \in V_{12}$.
- (ii) Each vertex of $G - V_{12}$ is mono-colored in G , and moreover, for each component H of $G - V_{12}$, all edges with at least one end in H have the same color as the edges of H .
- (iii) If $1 \leq |V_{12}| \leq 3$ and V_{12} is not a vertex cut, then the subgraph of G induced by V_{12} , $G[V_{12}]$, is connected.
- (iv) If $|V_{12}| = 1$ or 2 , then V_{12} is a vertex cut of G .

Let $u \in V(G)$ and

$$E_i(u) = \{e \in E_G(u) : f(e) = i\} \text{ for } i = 1, 2. \quad (4)$$

Proposition 2.3. (Shao, Proposition 2.4 of [4]) Each of the following holds:

- (i) $|X| = \sum_{u \in V_{12}} |E_1(u)| \cdot |E_2(u)|$.
- (ii) For each $u \in V_{12}$ and $i = 1, 2$, $|E_1(u)| \cdot |E_2(u)| \geq d_G(u) - 1$.
- (iii) If for each $u \in V_{12}$ and $i = 1, 2$, $|E_i(u)| \geq 2$, then $|E_1(u)| \cdot |E_2(u)| \geq 2(d_G(u) - 2)$.

A graph G is k -triangular if each edge of G lies in at least k triangles and G is triangular if it is 1-triangular. We summarize some properties of iterated line graphs of G in terms of $l(G)$ as follows.

Proposition 2.4. (Zhang, Eschen, Lai and Shao, Lemma 3.2 of [5]) Let G be a simple connected graph that is not a path, a cycle or $K_{1,3}$, with $l(G) = l \geq 1$. Then each of the following holds:

- (i) For an integer $m \geq 0$,

$$l(L^m(G)) = \begin{cases} l - m, & \text{if } 0 \leq m < l, \\ 1, & \text{if } m \geq l. \end{cases}$$

- (ii) For integers $k \geq 0$,

$$\delta(L^{l+k}(G)) \geq \begin{cases} 2, & \text{if } k = 0 \text{ or } k = 1, \\ 2^{k-2} + 2, & \text{if } k \geq 2. \end{cases}$$

- (iii) $L^l(G)$, $L^{l+1}(G)$ and $L^{l+2}(G)$ are triangular. Moreover, $L^{l+k}(G)$ is 2^{k-3} -triangular when $k \geq 3$.

- (iv) For an integer $k \geq 0$, $\kappa(L^{l+k}(G)) \geq k + 1$.

3. Proof of Theorem 1.2

In this section, we first prove Proposition 3.1, then apply induction on s to prove Theorem 1.2. Following the definitions of X , V_{12} and $E_i(u)$ (see (1), (3) and (4)) in Section 2, let X be a minimal essential edge cut of $L^{l+2}(G)$ and V_{12} the set of bi-colored vertices in $L^{l+1}(G)$. And $E_i(u)$ denotes the set of edges incident with u in $L^{l+1}(G)$ with color i . We use $d(v)$ for $d_{L^{l+1}(G)}(v)$. Now we prove Theorem 1.2(i) holds for $s = 2$.

Proposition 3.1. *Let G be a simple connected graph that is not a path, a cycle or $K_{1,3}$, with $l(G) = l$. Then $\kappa'_e(L^{l+2}(G)) \geq 6$.*

Proof. If $\delta(L^{l+1}(G)) \geq 3$, then by Theorem 1.1 and Proposition 1.3, $\kappa'_e(L^{l+2}(G)) \geq 2\kappa'_e(L^{l+1}(G)) - 2 = 2 \cdot 4 - 2 = 6$. Then Proposition 3.1 is proved.

Also, by Proposition 2.4(ii),

$$\delta(L^{l+1}(G)) \geq 2. \tag{5}$$

So we may assume that $\delta(L^{l+1}(G)) = 2$.

By Proposition 2.4(iv), $\kappa(L^{l+1}(G)) \geq 2$. By Proposition 2.2(iv), if $|V_{12}| = 1$, then V_{12} is a vertex cut of size one in $L^{l+1}(G)$, a contradiction. So $|V_{12}| \geq 2$.

Claim 1. If $|V_{12}| = 2$, then $|X| \geq 6$.

Proof of Claim 1. Let $V_{12} = \{v_1, v_2\}$. If $v_1v_2 \notin E(L^{l+1}(G))$, then as $L^{l+1}(G)$ is triangular (Proposition 2.4(iii)), and v_i is bi-colored, $|E_i(v_1)| \geq 2$ for $i = 1, 2$ by Proposition 2.2(ii). By Proposition 2.3(i) and (iii), $|X| \geq \sum_{i=1}^{|V_{12}|} 2(d(v_i) - 2) \geq 2 \cdot 2 + 2 \cdot 2 = 8$.

So we may assume $v_1v_2 \in E(L^{l+1}(G))$. By Proposition 2.2(iv), V_{12} is a vertex cut of $L^{l+1}(G)$. So we will finish the proof of Claim 1 by the following three cases.

Case 1 of Claim 1. $L^{l+1}(G) - V_{12}$ has at least two nontrivial components.

Proof of Case 1. Let G_1, G_2 be two nontrivial components of $L^{l+1}(G) - V_{12}$. By Proposition 1.3, $\kappa'_e(L^{l+1}(G)) \geq 4$. Then $||[V(G_i), V_{12}]| \geq 4$ for $i = 1, 2$, which implies $d(v_1) + d(v_2) \geq 4 + 4 + 2 = 10$. By Proposition 2.3(i) and (ii), $|X| \geq (d(v_1) - 1) + (d(v_2) - 1) = d(v_1) + d(v_2) - 2 \geq 10 - 2 = 8$. \square

Case 2 of Claim 1. $L^{l+1}(G) - V_{12}$ has at most one nontrivial component.

Proof of Case 2. Let G_1 be the nontrivial component and G_2 be a single vertex component (say u) of $L^{l+1}(G) - V_{12}$. Since $\kappa'_e(L^{l+1}(G)) \geq 4$ (Proposition 1.3), $||[V(G_1), V_{12}]| \geq 4$. Since u is a single vertex component of $L^{l+1}(G) - V_{12}$, by (5), we have $uv_1 \in E(L^{l+1}(G))$ and $uv_2 \in E(L^{l+1}(G))$. So $d(v_1) + d(v_2) \geq 4 + 2 + 2 = 8$. By Proposition 2.3(i) and (ii), $|X| \geq (d(v_1) - 1) + (d(v_2) - 1) = d(v_1) + d(v_2) - 2 \geq 8 - 2 = 6$. \square

Case 3 of Claim 1. Every component of $L^{l+1}(G) - V_{12}$ is trivial.

Then $L^{l+1}(G)$ is isomorphic to $K_{2,n-2}$ with an edge joining the two vertices of degree two, where n is the number of vertices of $L^{l+1}(G)$. If $n = 4$, then it has an essential 3-edge-cut, contrary to Proposition 1.3; if $n \geq 4$, then it contains an induced claw, contrary to the fact that line graphs are claw-free (see [2]). \square

Claim 2. If $|V_{12}| \geq 6$, then $|X| \geq 6$.

Proof of Claim 2. If $|V_{12}| \geq 6$, then by (5) and Proposition 2.3(i) and (ii), $|X| \geq \sum_{i=1}^{|V_{12}|} (d(v_i) - 1) \geq (2 - 1)|V_{12}| = 6$. \square

By Claims 1 and 2, we may assume that

$$3 \leq |V_{12}| \leq 5. \tag{6}$$

Claim 3. If the induced graph of V_{12} in $L^{l+1}(G)$ contains at least two independent edges, then $|X| \geq 8$.

Proof of Claim 3. We assume that $v_1, v_2, v_3, v_4 \in V_{12}$ such that $v_1v_2 \in E(L^{l+1}(G))$ and $v_3v_4 \in E(L^{l+1}(G))$. By Proposition 1.1, $d(v_1) + d(v_2) \geq \kappa'_e(L^{l+1}(G)) + 2$ and $d(v_3) + d(v_4) \geq \kappa'_e(L^{l+1}(G)) + 2$. By Proposition 2.3(i), (ii) and Proposition 1.3, $|X| \geq \sum_{i=1}^4 (d(v_i) - 1) = \sum_{i=1}^4 d(v_i) - 4 \geq 2(\kappa'_e(L^{l+1}(G)) + 2) - 4 = 2 \cdot (4 + 2) - 4 = 8$. This completes the proof of Claim 3. \square

By Claim 3, we may assume that

$$\text{the matching number of the induced graph of } V_{12} \text{ in } L^{l+1}(G) \text{ is at most one.} \tag{7}$$

Claim 4. If the induced graph of V_{12} in $L^{l+1}(G)$ has an isolated vertex, then $|X| \geq 6$.

Proof of Claim 4. Without loss of generality, we assume v_1 is an isolated vertex in the induced graph of V_{12} in $L^{l+1}(G)$. Since $L^{l+1}(G)$ is triangular (Proposition 2.4(iii)), and v_1 is bi-colored, $|E_i(v_1)| \geq 2$ for $i = 1, 2$ by Proposition 2.2(ii). By (6), $|V_{12}| \geq 3$. Together with Proposition 2.3(i), (ii), (iii) and (5), we have $|X| = \sum_{u \in V_{12}} |E_1(u)| \cdot |E_2(u)| \geq 2(d(v_1) - 2) + (2 - 1) \cdot (|V_{12}| - 1) \geq 2 \cdot 2 + (3 - 1) = 6$. \square

By Claim 4, we may assume

$$\text{the induced graph of } V_{12} \text{ in } L^{l+1}(G) \text{ has no isolated vertices.} \tag{8}$$

Claim 5. The induced graph of V_{12} in $L^{l+1}(G)$ is isomorphic to K_3 or $K_{1,2}$.

Proof of Claim 5. By (7) and (8), the induced graph of V_{12} in $L^{l+1}(G)$ is isomorphic to K_3 or $K_{1,|V_{12}|-1}$. If $4 \leq |V_{12}| \leq 5$, then $K_{1,|V_{12}|-1}$ contains a claw, contrary to the fact that $L^{l+1}(G)$ is claw-free. So by (6), $|V_{12}| = 3$ and by (7) and (8), the induced graph of V_{12} in $L^{l+1}(G)$ is isomorphic to K_3 or $K_{1,2}$. \square

Claim 6. If each vertex in the induced graph of V_{12} in $L^{l+1}(G)$ has degree at least three, then $|X| \geq 6$.

Proof of Claim 6. Let $V_{12} = \{v_1, v_2, v_3\}$ and assume $d(v_i) \geq 3$ for $i = 1, 2, 3$. By Proposition 2.3(i), (ii) and (6), $|X| \geq \sum_{i=1}^3 (d(v_i) - 1) \geq (3 - 1)|V_{12}| \geq 6$. \square

By Claim 6, we assume

$$\text{at least one vertex in the induced graph of } V_{12} \text{ in } L^{l+1}(G) \text{ has degree two.} \tag{9}$$

Now we will finish the proof of Theorem 1.2 by the following two cases.

Case 1. $L^{l+1}(G) - V_{12}$ is connected.

Proof of Case 1. Let G' be the graph induced by $L^{l+1}(G) - V_{12}$. By Proposition 2.2(ii), we assume all edges in $[V(G'), V_{12}]$ have color 1. As v_1, v_2, v_3 are bi-colored, if the induced graph of V_{12} in $L^{l+1}(G)$ is $K_{1,2}$, then both edges of $K_{1,2}$ must have color 2; if the induced graph of V_{12} in $L^{l+1}(G)$ is K_3 , then by Proposition 2.1(i), at least two edges of K_3 have color 2. In

either case, without loss of generality we may assume that both v_1v_2 and v_2v_3 have color 2, which means $|E_2(v_2)| \geq 2$. Since v_2 is bi-colored, there is an edge in $[V(G'), \{v_2\}]$ with color 1, which implies that the degree of v_2 is at least three. By (9), we may assume $d(v_1) = 2$, then $d(v_2) \geq 4$ by Propositions 1.1 and 1.3. Then $|[V(G'), \{v_2\}]| \geq 2$ and so $|E_1(v_2)| \geq 2$. Together with $|E_2(v_2)| \geq 2$, by Proposition 2.3(i), (ii) and (iii), $|X| \geq 2(d(v_2) - 2) + (d(v_1) - 1) + (d(v_3) - 1) \geq 2 \cdot 2 + (2 - 1) + (2 - 1) = 6$. \square

Case 2. $L^{l+1}(G) - V_{12}$ is disconnected.

By Claim 5, we only need to prove Case 2 by the following two cases.

Case 2.1. The induced graph of V_{12} in $L^{l+1}(G)$ is isomorphic to K_3 .

Proof of Case 2.1. By (9), without loss of generality, we assume that $d(v_2) = 2$, then by Propositions 1.1 and 1.3, $d(v_1) \geq 4$ and $d(v_3) \geq 4$. By Proposition 2.3(i) and (ii), $|X| \geq (d(v_1) - 1) + (d(v_2) - 1) + (d(v_3) - 1) \geq (4 - 1) + (2 - 1) + (4 - 1) = 7$. \square

Case 2.2. The induced graph of V_{12} in $L^{l+1}(G)$ is isomorphic to $K_{1,2}$.

Proof of Case 2.2. Assume $v_1v_2 \in E(L^{l+1}(G))$ and $v_2v_3 \in E(L^{l+1}(G))$. If $d(v_2) = 2$, then by Propositions 1.1 and 1.3, $d(v_1) \geq 4$ and $d(v_3) \geq 4$. Using a similar argument to Case 2.1, we have $|X| \geq 7$.

By (9), we may assume that $d(v_1) = 2$.

Then by Propositions 1.1 and 1.3 again, $d(v_2) \geq 4$. If $d(v_3) \geq 3$, then $|X| \geq (d(v_1) - 1) + (d(v_2) - 1) + (d(v_3) - 1) \geq (2 - 1) + (4 - 1) + (3 - 1) = 6$. So we assume $d(v_3) = 2$.

If $d(v_2) \geq 5$, then $|X| \geq (d(v_1) - 1) + (d(v_2) - 1) + (d(v_3) - 1) \geq (2 - 1) + (5 - 1) + (2 - 1) = 6$. So we assume $d(v_2) = 4$.

Now we have $d(v_1) = d(v_3) = 2$ and $d(v_2) = 4$. Since $L^{l+1}(G)$ is triangular, we assume v_1v_2, v_2v_3 lie in triangles T_1, T_2 , respectively. If T_1, T_2 share a common edge, then $L^{l+1}(G)$ must be the graph depicted in Figure 3(a), where H represents a connected subgraph containing the edge u_1u_2 ; if T_1, T_2 do not share a common edge, then $L^{l+1}(G)$ must be the graphs depicted in Figure 3(b) and 3(c), where H in 2(b) represents a connected subgraph containing the vertices u_1, u_2 and two circles in 2(c) represent either two isolated vertices u_1, u_2 or two disjoint connected subgraphs containing u_1, u_2 , respectively. In Figure 3(a) and 3(b), since H is a connected subgraph, $L^{l+1}(G) - V_{12}$ is connected, contrary to the assumption of Case 2; in Figure 3(c), $L^{l+1}(G)$ has an essential 2-edge-cut, contrary to Proposition 1.3.

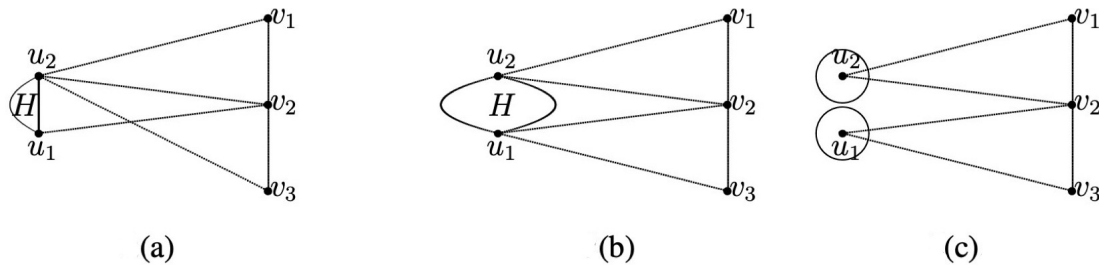


Figure 3. Iterated Line graph $L^{l+1}(G)$.

This completes Case 2.2. \square

Hence, Proposition 3.1 is established. □

Next we will use Proposition 3.1 to prove Theorem 1.2.

Proof of Theorem 1.2(i). The case $s = 1$ is implied by Proposition 1.3 and the case $s = 2$ is implied by Proposition 3.1. We assume $s \geq 3$.

By Proposition 2.4(ii), for $s \geq 3$, $\delta(L^{l+s-1}(G)) \geq 3$. Now we prove (i) by induction. Assume that $\kappa'_e(L^{l+s-1}(G)) \geq 2^{s-1} + 2$. By Theorem 1.1, $\kappa'_e(L^{l+s}(G)) \geq 2\kappa'_e(L^{l+s-1}(G)) - 2 \geq 2(2^{s-1} + 2) - 2 = 2^s + 2$. Then (i) is proved. □

Proof of Theorem 1.2(ii). By (i) and Proposition 1.2(ii), for $s \geq 2$, $\kappa(L^{l+s}(G)) = \kappa'_e(L^{l+s-1}(G)) \geq 2^{s-1} + 2$. □

Next we give an example to show that both bounds in Theorem 1.2 are best possible. Let G be the graph depicted in Figure 4(a). Then $l(G) = 2$. The first three iterated line graphs of G are depicted in Figure 4(b)-(d). Note that $L^3(G)$ has a 4-cycle in which every vertex has degree equal to 3. By the definition of line graphs, a 4-cycle of a graph generates a 4-cycle in its line graph, and the degree of a vertex in a line graph is the degree sum of the end vertices of the corresponding edge in the original graph minus 2. So $L^4(G)$ has a 4-cycle in which every vertex has degree equal to $2 \cdot 3 - 2 = 4$. In general, for $s \geq 2$, $L^{l+s}(G)$ has a 4-cycle in which every vertex has degree equal to $2^{s-1} + 2$. Let $uv \in E(L^{l+s}(G))$ be an edge on the 4-cycle such that both u and v have degree equal to $2^{s-1} + 2$. Then $E_{L^{l+s}(G)}(u) \cup E_{L^{l+s}(G)}(v) - \{uv\}$ is an essential edge cut of size $2 \cdot (2^{s-1} + 2) - 2 = 2^s + 2$. Note that $\kappa'_e(L^{l+1}(G)) = \kappa'_e(L^3(G)) = 4$. Hence $\kappa'_e(L^{l+s}(G)) = 2^s + 2$ for $s \geq 1$. By Proposition 1.2(ii), $\kappa(L^{l+s}(G)) = \kappa'_e(L^{l+s-1}(G)) = 2^{s-1} + 2$ for $s \geq 2$.

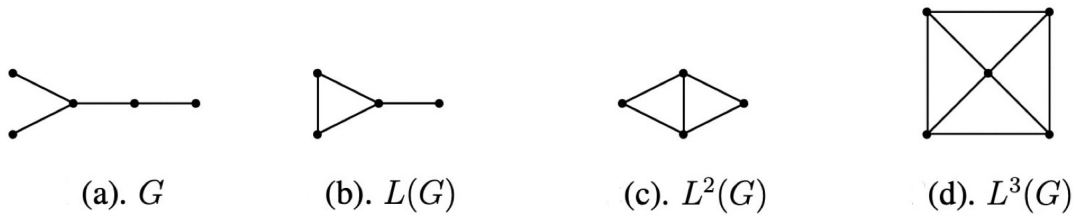


Figure 4. An example for the sharpness of the bounds.

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