On the domination and signed domination numbers of zero-divisor graph

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Abstract

Let $R$ be a commutative ring (with 1) and let $Z(R)$ be its set of zero-divisors. The zero-divisor graph $\Gamma(R)$ has vertex set $Z^*(R) = Z(R) \setminus \{0\}$ and for distinct $x, y \in Z^*(R)$, the vertices $x$ and $y$ are adjacent if and only if $xy = 0$. In this paper, we consider the domination number and signed domination number on zero-divisor graph $\Gamma(R)$ of commutative ring $R$ such that for every $0 \neq x \in Z^*(R)$, $x^2 \neq 0$. We characterize $\Gamma(R)$ whose $\gamma(\Gamma(R)) + \gamma_{s}(\Gamma(R)) \in \{n + 1, n, n - 1\}$, where $|Z^*(R)| = n$.

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1. Introduction

The study on graphs from algebraic structures is an interesting subject for mathematician. In recent years, many algebraists as well as graph theorists have focused on the zero-divisor graph of rings. In [1], Anderson and Livingston introduced the zero-divisor graph of a commutative ring $R$ with identity, denoted by $\Gamma(R)$, as the graph with vertices $Z^*(R) = Z(R) \setminus \{0\}$, the set of nonzero zero-divisors of $R$, and for distinct vertices $x$ and $y$ are adjacent if and only if $xy = 0$. A dominating set for $\Gamma$ is a subset $D$ of $V$ such that every vertex not in $D$ is adjacent to at least
one member of \( D \). The \textit{domination number} is the number of vertices in a smallest dominating set for \( \Gamma \) and denoted by \( \gamma(\Gamma) \). Oystein Ore introduced the terms “dominating set” and ”domination number” in [10] and has proved if \( \Gamma \) has \( n \) vertices and no isolated vertices, then \( \gamma(\Gamma) \leq \frac{n}{2} \).

For a vertex \( v \in V(\Gamma) \), the closed neighborhood \( N[v] \) of \( v \) is the set consisting of \( v \) and all of its neighbors. For a function \( g : V(\Gamma) \longrightarrow \{-1, 1\} \) and a vertex \( v \in V \) we define \( g[v] = \sum_{u \in N[v]} g(u) \). A \textit{signed dominating function} of \( \Gamma \) is a function \( g : V(\Gamma) \longrightarrow \{-1, 1\} \) such that \( g[v] \geq 0 \) for all \( v \in V(\Gamma) \). The \textit{weight} of a function \( g \) is \( \omega(g) = \sum_{v \in V(\Gamma)} g(v) \). The \textit{signed domination number} \( \gamma_s(\Gamma) \) is the minimum weight of a signed dominating function on \( \Gamma \). A signed dominating function of weight \( \gamma_s(\Gamma) \) is called a \( \gamma_s(\Gamma) \)–function. This concept was defined in [3] and has been studied by several authors (see for instance \([4, 7, 8, 13, 14]\)). For a graph \( \Gamma \) the set of all vertices of \( \Gamma \) is denoted by \( V(\Gamma) \). If \( \Gamma \) is a graph, then the \textit{complement} of \( \Gamma \), denoted by \( \overline{\Gamma} \) is a graph with vertex set \( V(\Gamma) \) in which two vertices are adjacent if and only if they are not adjacent in \( \Gamma \). A graph is said to be \textit{connected} if each pair of vertices are joined by a walk. The number of edges in a shortest walk joining \( v_i \) and \( v_j \) is called the \textit{distance} between \( v_i \) and \( v_j \) and denoted by \( d(v_i, v_j) \). The maximum value of the distance function in a connected graph \( \Gamma \) is called the \textit{diameter} of \( \Gamma \) and denoted by \( \text{diam}(\Gamma) \). The \textit{complete graph} \( K_n \) is the graph with \( n \) vertices in which each pair of vertices are adjacent. The \textit{corona} \( \Gamma_1 \circ \Gamma_2 \) is the graph formed by one copy of \( \Gamma_1 \) and \( |V(\Gamma_1)| \) copies of \( \Gamma_2 \) where the \( i \text{th} \) vertex of \( \Gamma_1 \) is adjacent to every vertex in the \( i \text{th} \) copy of \( \Gamma_2 \).

In this work, we consider the domination and signed domination number on zero-divisor graph \( \Gamma(R) \) for commutative ring \( R \). The main results are in the following.

**Theorem 1.1.** \( \gamma_s(\Gamma(R)) = n \) if and only if \( \Gamma(R) \) is isomorphic to \( K_{1, n-1} \) or \( K_3 \circ K_1 \).

**Theorem 1.2.** Let \( |R| \) be odd. Then \( \gamma_s(\Gamma(R)) = n - 2 \) if and only if \( \Gamma(R) \) is a cycle \( C_4 \).

**Theorem 1.3.** \( \gamma(\Gamma(R)) + \gamma(\overline{\Gamma(R)}) = n \) if and only if \( \Gamma(R) \) is a cycle \( C_4 \) or a path \( P_3 \).

**Theorem 1.4.** \( \gamma(\Gamma(R)) + \gamma(\overline{\Gamma(R)}) = n - 1 \) if and only if \( \Gamma(R) \) is isomorphic to a \( K_{1, 3} \) or a \( K_3 \circ K_1 \).

2. Preliminaries

First we give some facts that are needed in the next sections.

**Theorem 2.1.** [1] Let \( R \) be a commutative ring. Then \( \Gamma(R) \) is connected and \( \text{diam}(\Gamma(R)) \leq 3 \). Moreover, if \( \Gamma(R) \) contains a cycle, then \( \text{girth}(\Gamma(R)) \leq 7 \).

**Theorem 2.2.** [1] Let \( R \) be a finite commutative ring with \( |\Gamma(R)| \geq 4 \). Then \( \Gamma(R) \) is a star graph if and only if \( R = \mathbb{Z}_2 \times F \) where \( F \) is a finite field. In particular, if \( \Gamma(R) \) is a star graph, then \( |\Gamma(R)| = p^n \) for some prime \( p \) and \( n \geq 0 \). Conversely, each star graph of order \( p \) can be realized as \( \Gamma(R) \).

**Theorem 2.3.** [10] If a graph \( \Gamma \) has \( n \) vertices and no isolated vertices, then \( \gamma(\Gamma) \leq \frac{n}{2} \).
Theorem 2.4. [9] For any graph $\Gamma$ with $n$ vertices:

i. $\gamma(\Gamma) + \gamma(\overline{\Gamma}) \leq n + 1$.

ii. $\gamma(\Gamma)\gamma(\overline{\Gamma}) \leq n$.

Theorem 2.5. [11][5] For a graph $\Gamma$ with even order $n$ and no isolated vertices, $\gamma(\Gamma) = \frac{n}{2}$ if and only if the components of $\Gamma$ are the cycle $C_4$ or the corona $H \circ K_1$ where $H$ is a connected graph.

Lemma 2.1. [8] Let $\Gamma$ be a complete graph of order $n$, then

$$\gamma_s(\Gamma) = \begin{cases} 1 & n \text{ is odd} \\ 2 & n \text{ is even} \end{cases}$$

Theorem 2.6. [8] Let $\Gamma$ be a graph with $n$ vertices, then

i. $\gamma_s(\Gamma) + \gamma_s(\overline{\Gamma}) = 2n$ and $\gamma_s(\Gamma)\gamma_s(\overline{\Gamma}) = n^2$ if and only if $\Gamma \in \{P_1, P_2, P_3, \overline{P}_3, P_4\}$, where $P_i$ is a path on $i$ vertices.

ii. $\gamma_s(\Gamma) + \gamma_s(\overline{\Gamma}) = 2n - 2$ and $\gamma_s(\Gamma)\gamma_s(\overline{\Gamma}) = n^2 - 2n$ for exactly 12 graph in Figure 1.

![Graphs G1 to G12](image-url)

Figure 1. $\gamma_s(\Gamma) + \gamma_s(\overline{\Gamma}) = 2n - 2$ and $\gamma_s(\Gamma)\gamma_s(\overline{\Gamma}) = n^2 - 2n$.

Lemma 2.2. [8] A graph $\Gamma$ has $\gamma_s(\Gamma) = n$ if and only if every $v \in \Gamma$ is either isolated, an endvertex or adjacent to an endvertex.
3. Signed domination number on zero-divisor graph

Throughout this paper, $R$ is a commutative ring such that $|Z^*(R)| = n$ and for every non-zero element $x$, $x^2 \neq 0$. Also $\overline{\Gamma}(R)$ denotes the complement graph of the zero-divisor graph on $R$.

Lemma 3.1. The cycle $C_n$ is a zero-divisor graph of a ring if and only if $n = 4$.

Proof. Let $\Gamma(R)$ be the zero-divisor graph of a commutative ring $R$. Since $girth(\Gamma(R)) \leq 7$, then $n \leq 7$. On the contrary, let $\Gamma(R) \simeq C_n$ and $n \geq 5$ or $n = 3$. If $n \geq 5$, then $a_1 - a_2 - \ldots - a_n - a_1$. So $a_1 + a_3 \in \text{ann}(a_2) = \{0, a_1, a_3\}$ and so $a_1 + a_3 = 0$. Thus $a_4a_1 = 0$. This is impossible. Let $\Gamma(R)$ be $K_3$. Then $Z(R) = \{0, a, b, c\}$. So $\text{ann}(a) = \{0, b, c\}$ and $\text{ann}(b) = \{0, a, c\}$. Thus $b = -c = a$. This is a contradiction. Conversely, the zero divisor graph of ring $Z_3 \times Z_3$ is a cycle $C_4$. \qed

Proof of Theorem 1.1. Let $\gamma_s(\Gamma(R)) = n$. Since $\Gamma(R)$ is a connected graph, by Lemma 2.2, every vertex is an endvertex or adjacent to an end-vertex. If $x \in Z^*(R)$ and $\deg(x) = 1$, then $\text{ann}(x) = \{0, y\}$ where $xy = 0$. So $O(y) = 2$ in group $(R, +)$. Hence $|R|$ has even order. Let $A = \{a; \deg(a) > 1\}$. Since $\text{diam}(\Gamma(R)) \leq 3$, the induced subgraph on $A$ is a complete graph. Consider four cases:

Case 1. If $|A| = 1$, then $\Gamma(R)$ is $K_{1,n-1}$.

Case 2. Let $A = \{a, b\}$. Then $\text{ann}(a) \cap \text{ann}(b) = \{0\}$. Suppose that $u \in \text{ann}(a)$ and $v \in \text{ann}(b)$. Since $\deg(a), \deg(b) > 1$, then $\deg(u) = \deg(v) = 1$ and also $uva = uvb = 0$. Hence, $uv \in \text{ann}(a) \cap \text{ann}(b)$ and so $uv = 0$. This is a contradiction by $\deg(u) = \deg(v) = 1$.

Case 3. Let $A = \{a, b, c\}$. Let $E(a)$ be the set of endvertex adjacent to $a$. Since $b, c \in \text{ann}(a)$ and $O(a) = O(b) = 2$, $\text{ann}(a)$ is a subgroup of $(R, +)$ of even order. Hence $\text{odd}(a)$ is odd. The same conclusion can be drawn for $b, c$. We claim that $|E(a)| = 1$. On the contrary, suppose that $|E(a)| \geq 3$. There is no loss of generality in assuming $E(a) = \{x_1, x_2, x_3\}$. So $\text{ann}(a) = \{0, b, c, x_1, x_2, x_3\}$. Hence $x_1 = -x_3$ and $O(x_2) = 2$ or $O(x_1) = 2$ for $i \in \{1, 2, 3\}$. In the both cases, $x_1 + x_2 + x_2 \neq x_3$. Let $y \in E(b)$. Then $x_1ya = x_1yb = 0$. So $x_1y \in \text{ann}(a) \cap \text{ann}(b) = \{0, c\}$. Since $\deg(y) = 1$, $x_1y = c$. In the same manner we can see that $x_2y = x_3y = c$. Hence $y(x_1 + x_2) = y(x_2 + x_3) = 2c = 0$. Thus $x_1 + x_2, x_2 + x_3 \in \text{ann}(y) = \{0, b\}$. So $x_1 + x_2 = x_2 + x_3 = b$ and so $x_1 = x_3$. This is a contradiction. Therefore $|E(a)| = |E(b)| = |E(c)| = 1$ and $\Gamma(R)$ is $K_3 \circ K_1$.

Case 4. Let $A = \{a_1, \ldots, a_t\}$ and $t > 3$. Then $\text{ann}(a_i) = \{0, a_1, \ldots, \hat{a_i}, \ldots, a_t\} \cup E(a_i)$ for $i \in \{1, \ldots, t\}$. So $\bigcap_{i=1}^{t-1} \text{ann}(a_i) = \{0, a_{t-1}, a_t\}$. Hence $a_{t-1} = -a_t$. Since $N(a_{t-1}) \neq N(a_t)$, this is impossible. \qed

Corollary 3.1. If $\gamma_s(\Gamma(R)) = n$, then $\gamma_s(\overline{\Gamma}(R)) \in \{0, 3\}$.
Proof. By Theorem 1.1, $\Gamma(R) \simeq K_{1,n-1}$ or $K_3 \circ K_1$. If $\Gamma(R) \simeq K_{1,n-1}$, then $\overline{\Gamma(R)}$ is $K_1 \cup K_{n-1}$. Since $|Z(R)|$ is even, then $n$ is odd and so $\gamma_s(K_{n-1}) = 2$ and $\gamma_s(\overline{\Gamma(R)}) = 3$. If $\Gamma(R) \simeq K_3 \circ K_1$, then $\overline{\Gamma(R)}$ is the graph in Figure 2. Let $V_1 = \{x, y, z\}$ and $V_2 = \{a, b, c\}$. Define $f : V(\overline{\Gamma(R)}) \rightarrow \{-1, +1\}$ such that

$$f(u) = \begin{cases} 
-1 & u \in V_1; \\
+1 & u \in V_2.
\end{cases}$$

It is clear that $f$ is a signed dominating function and $\omega(f) = 0$. If $g$ is a function such that $\omega(g) < 0$, then $g$ is not a signed dominating function. Therefore $\gamma_s(\overline{\Gamma(R)}) = 0$. \hfill \Box

Corollary 3.2. If $\gamma_s(\Gamma(R)) = n$, then $|R| \in \{2^k, 2p^k\}$ where $p$ is prime.

Proof. By Theorem 1.1, $\Gamma(R) \simeq K_{1,n-1}$ or $K_3 \circ K_1$. If $\Gamma(R) \simeq K_{1,n-1}$, then by Theorem 2.2, $R \simeq Z_2 \times F$ where $F$ is a finite field. So $|R| = 2p^k$. Let $\Gamma(R) \simeq K_3 \circ K_1$. Let $V(\Gamma(R)) = \{a_i, x_i; \deg(x_i) = 1, \deg(a_i) = 3, 1 \leq i \leq 3\}$. So $|R|$ is even. If $p \mid |R|$ ($p$ is odd prime number), then there is $0 \neq r \in R$ such that $O(r) = p$. Hence $pr = 0$. Also $(p-1)a_i = 0$. Thus $ra_i = r(pa_i) = 0$. So $r \in \text{ann}(a_i)$ for every $1 \leq i \leq 3$. Hence $r = 0$. This is a contradiction. Therefore $|R| = 2^k$. \hfill \Box

The Proof of Theorem 1.2 Since $|R|$ is odd, $\delta \geq 2$. Let $x \in R$ and $\deg(x) = 2k + 1$. Then $|\text{ann}(x)| = 2k + 2$. This is a contradiction by $|R|$ is odd. So all vertices have even degree. Since $\text{diam}(\Gamma(R)) \leq 3$, there are three cases:

Case 1. If $\text{diam}(\Gamma(R)) = 1$, then $\Gamma(R)$ is complete graph $K_n$. Since all vertices have even degree, $n$ is odd and so $\gamma_s(\Gamma(R)) = 1$. Hence $n = 3$ and $\Gamma(R)$ is $K_3$. This is impossible by Lemma 3.1.

Case 2. If $\text{diam}(\gamma(R)) = 3$, then there are $a, b \in Z^*(R)$ such that $d(a, b) = 3$. Define signed dominating function $f : V(\Gamma(R)) \rightarrow \{-1, +1\}$ such that $f(a) = f(b) = -1$ and $f(x) = 1$ for $x \in Z^*(R) \setminus \{a, b\}$. Thus $\gamma_s(\Gamma(R)) < n - 2$. This is impossible.
Case 3. Let diam($\Gamma(R)$) = 2. If $\Delta = 2$, then $\Gamma(R)$ is a cycle. So $\Gamma(R) \simeq C_4$, by Theorem 3.1. Let $deg(y) = \Delta \geq 4$. Let $ann(y) = \{0, a_1, \ldots, a_t\}$ where $t$ is even and $t \geq 4$. So $O(a_i) \neq 2$.

Hence, $-a_i \in ann(y)$. Thus $ann(y) = \{0, a_1, -a_1, \ldots, a_t, -a_t\}$. Let $x \in N(a_1)$. If there is $2 \leq j \leq \frac{t}{2}$ such that $\{a_1, a_j\} \notin E(\Gamma(R))$, then $d(x, a_j) > 2$. Otherwise, there is $z \in N(a_j) \setminus ann(y)$ and so $d(x, z) = 3$. This is not true. So for every $x \in N(a_1)$, $\text{deg}(x) \geq 4$. Define $f : V(\Gamma(R)) \rightarrow \{-1, +1\}$ such that $f(a_1) = f(-a_1) = -1$ and $f(v) = 1$ for every $v \in V(\Gamma(R)) \setminus \{a_1, -a_1\}$. So $f$ is a signed dominating function and so $\gamma(\Gamma(R)) < n - 2$. This is a contradiction. \qed

**Theorem 3.1.** If $\gamma_s(\Gamma(R)) + \gamma_s(\overline{\Gamma(R)}) = 2n$, then $|R| \in \{2^k, 2 \times 3^k\}$.

**Proof.** Since $\Gamma(R)$ is a connected graph, by Theorem 2.6, $\Gamma(R)$ is one of the paths in $\{P_1, P_2, P_3, P_4\}$. It is known $P_4$ is not a zero-divisor graph.

If $\Gamma(R)$ is $P_1$, then $Z(R) = \{0, x\}$. So $x^2 = 0$. This is impossible.

Let $\Gamma(R)$ be $P_2$. Then $Z(R) = \{0, a, b\}$ and $O(a) = O(b) = 2$. So $|R|$ is even. If $p \mid |R|$ where $p$ is an odd prime number, then there is $r \in R$ such that $O(r) = p$. Hence $(p - 1)a = 0$. Thus $ra = r(pa) = 0$. So $r \in ann(a)$ and so $r = b$. This is a contradiction. If $\Gamma(R)$ is $a - c - b$, then $ann(c) = \{0, a, b\}$. So $b = -a$ and so $O(a) = 3$. Also $O(c) = 2$. Also by Theorem 2.2, $R \simeq Z_2 \times F$. So $|R| = 2 \times 3^k$. \qed

**Theorem 3.2.** If $\gamma_s(\Gamma(R)) + \gamma_s(\overline{\Gamma(R)}) = 2n - 2$, then $|R| = 2p^k$ where $p$ is an odd prime.

**Proof.** By Theorem 2.6 and Lemma 3.1 and since $\Gamma(R)$ is a connected graph, $\Gamma(R) \in \{K_{1,3}, K_{1,4}, G_1, G_2\}$ where $G_1, G_2$ are two graphs in Figure 3. We show that $G_1$ and $G_2$ are not a zero-divisor graph. If $G_1$ is a zero-divisor graph, then $b(a + e) = 0$. So $a + e \in ann(b) = \{0, a, e\}$. Hence $e = -a$. This is a contradiction by $c, d \notin ann(a)$. Similar argument applies for $G_2$.

If $\Gamma(R)$ is $K_{1,3}$ or $K_{1,4}$, then likewise Corollary 3.2, $|R| = 2p^k$. \qed

**Figure 3.** $G_1$ and $G_2$ in Theorem 3.2.

4. Domination number on zero-divisor graph

**Theorem 4.1.** $\gamma(\Gamma(R)) = \frac{n}{2}$ if and only if $\Gamma(R)$ is a cycle $C_4$ or a $K_3 \circ K_1$. 

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Case 1. Let $\gamma(\Gamma(R)) = \frac{n}{2}$. By Theorem 2.5, $\Gamma(R)$ is the a cycle $C_4$ or the corona $H \circ K_1$ where $H$ is a connected graph. If $\Gamma(R)$ is not $C_4$, then $\Gamma(R) \simeq H \circ K_1$. Let $A = \{a_i; \deg(a_i) > 1\}$. Since $\text{diam}(\Gamma(R)) \leq 3$, the induced subgraph on $A$ is complete. If $|A| = 2$, then $\Gamma(R)$ is a path $P_4$. This is impossible. If $|A| > 3$, then $\bigcap_{i=1}^{t-2} \text{ann}(a_i) = \{0, a_{t-1}, a_t\}$. Hence $a_t = -a_{t-1}$. This is a contradiction. So $|A| = 3$ and so $\Gamma(R) \simeq K_3 \circ K_1$. The converse is clear. \qed

**Theorem 4.2.** $\gamma(\Gamma(R)) + \gamma(\Gamma(R)) = n + 1$ if and only if $\Gamma(R)$ is complete graph $K_n$.

**Proof.** Let $\gamma(\Gamma(R)) + \gamma(\Gamma(R)) = n + 1$. By Theorem 2.3, $\gamma(\Gamma(R)) \leq \frac{n}{2}$. So $\gamma(\Gamma(R)) > \frac{n}{2}$ and so $\Gamma(R)$ has isolated vertex. Hence $\gamma(\Gamma(R)) = 1$ and $\gamma(\Gamma(R)) = n$. Thus all vertices of $\Gamma(R)$ are isolated. Therefore $\Gamma(R) \simeq K_n$. \qed

**Proof of Theorem 1.3.** Let $\gamma(\Gamma(R)) + \gamma(\Gamma(R)) = n$. Since $\Gamma(R)$ is a connected graph, $\gamma(\Gamma(R)) \leq \frac{n}{2}$. We consider following cases:

Case 1. Let $\gamma(\Gamma(R)) = \frac{n}{2}$. By Theorem 4.1 and above equality, $\Gamma(R)$ is a $C_4$.

Case 2. If $\gamma(\Gamma(R)) < \frac{n}{2}$, then $\gamma(\Gamma(R)) > \frac{n}{2}$. So $\Gamma(R)$ has an isolated vertex and so $\gamma(\Gamma(R)) = 1$.

Also $\gamma(\Gamma(R)) = n - 1$. Thus $\Gamma(R)$ is $P_2 \cup (n - 2)K_1$. It is clear that $n \geq 3$.

Sub case I. If $n > 3$, then likewise the proof of Theorem 4.1, the contradiction reaches.

Sub case II. If $n = 3$, then $\Gamma(R) \simeq P_2 \cup K_1$. So $\Gamma(R)$ is the path $P_3$.

The converse is easy. \qed

**Proof of Theorem 1.4.** Let $\gamma(\Gamma(R)) + \gamma(\Gamma(R)) = n - 1$. Since $\Gamma(R)$ has no isolated vertices, $\gamma(\Gamma(R)) \leq \frac{n}{2}$. There are three cases:

Case 1. If $\gamma(\Gamma(R)) = \frac{n}{2}$, then $\Gamma(R)$ is $K_3 \circ K_1$ or $C_4$ by Theorem 4.1. But $K_3 \circ K_1$ is not satisfied in $\gamma(\Gamma(R)) + \gamma(\Gamma(R)) = n - 1$.

Case 2. Let $\gamma(\Gamma(R)) = \frac{n}{2} - 1$. Then $\gamma(\Gamma(R)) = \frac{n}{2}$. By Theorem 2.4, $0 \leq n \leq 6$. So $n \in \{4, 6\}$.

Sub case I. Let $n = 4$. Then $\gamma(\Gamma(R)) = 1$ and $\gamma(\Gamma(R)) = 2$. So $\Gamma(R)$ is $K_1, G$ or $G$ in Figure 4. Let $G$ be a zero-divisor graph. Since $\deg(a) = 1$, $O(b) = 2$. On the other hand, $\text{ann}(c) = \{0, b, d\}$. So $d = -b$. This is not true.

Sub case II. If $n = 6$, then $\gamma(\Gamma(R)) = 2$ and $\gamma(\Gamma(R)) = 3$. So $\Gamma(R)$ is a graph without isolated vertex. Hence by Theorem 2.5, $\Gamma(R)$ is $C_4 \cup P_2, 3P_2$ or $K_3 \circ K_1$. So $\Gamma(R)$ is $G_1, G_2$ and $G_3$ in Figure 4, respectively. In graph $G_1$, $c(d + e) = 0$ and so $d + e \in \text{ann}(c)$. Hence $d + e = 0$ or $f$. Thus $ad = 0$ or $bd = 0$. This is a contradiction. In graph $G_2$, $d + f \in \text{ann}(a)$. But all cases are impossible. In graph $G_3$, Since $b(d + f) = 0$, $d = -f$. So $cf = 0$. This is not true.
Case 3. If $\gamma(\Gamma(R)) < \frac{n}{2} - 1$, then $\Gamma(R)$ has an isolated vertex. So $\gamma(\Gamma(R)) = 1$ and so $\gamma(\Gamma(R)) = n - 2$. Hence $\Gamma(R)$ is $P_3 \cup (n - 3)K_1$ or $K_3 \cup (n - 3)K_1$. If $n = 4$, then $\Gamma(R)$ is $G$ in Figure 4 or $K_{1,3}$ respectively. But $G$ is not a zero-divisor graph of a ring. For $n > 4$, the contradiction reached by the same method in Theorem 4.1.

![Diagram](https://example.com/diagram)

Figure 4. $\Gamma(R)$ in the proof of Theorem 1.4, Cases 2 and 3.

References


