A note on edge-disjoint contractible Hamiltonian cycles in polyhedral maps

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Abstract

We present a necessary and sufficient condition for the existence of edge-disjoint contractible Hamiltonian cycles in the edge graph of polyhedral maps.

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1. Introduction and Definitions

Recall the following definitions (see Maity and Upadhyay [7]) that a graph $G := (V, E)$ is a simple graph with vertex set $V$ and edge set $E$. A surface $S$ is a connected, compact, 2-dimensional manifold without boundary. A map $M$ on a surface $S$ is an embedding of a finite graph $G$ such that the closure of components of $S \setminus G$ is $p$-gonal 2-disc for $p \geq 3$. The components are also called facets. The map $M$ is called a polyhedral map if nonempty intersection of any two facets of the map is either a vertex or an edge. We call $G$ the edge graph of the map and denote it by $EG(M)$. The vertices and edges of $G$ are also called vertices and edges of the map, respectively. A path $P$ in a graph $G$ is a subgraph $P : [v_1v_2 \ldots v_n]$ of $G$, such that the vertex set of $P$ is $V(P) = \{v_1, v_2, \ldots, v_n\} \subseteq V(G)$ and $v_iv_{i+1}$ are edges in $P$ for $1 \leq i \leq n - 1$. A path $P : [v_1, v_2, \ldots, v_n]$ in $G$ is said to be a cycle if $v_nv_1$ is also an edge in $P$. A graph without any cycle is called a tree.

Let $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$ be two graphs. Then $G_1 \cup G_2$ is defined to be a graph $G(V, E)$ for
which $V = V_1 \cup V_2$ and $E = E_1 \cup E_2$. In this case $G$ is called union of the graphs $G_1$ and $G_2$. Similarly, $G_1 \cap G_2$ is the graph $G(V, E)$ for which $V = V_1 \cap V_2$ and $E = E_1 \cap E_2$. In this case $G$ is called intersection of $G_1$ and $G_2$. These definitions remain valid for a finite number of graphs as well. See Mohar and Thomassen [8] for details about graphs on surfaces and Bondy and Murthy [1] for terminology related to graph theory.

In this note we are interested in finding out whether edge-disjoint Hamiltonian cycles exist in the edge graph of a polyhedral map. Such cycles in graphs have been studied previously. For example, Nash-Williams [10] generalised a result of Dirac [4] about existence of Hamiltonian cycles and showed that every graph on $n$ vertices contains at least $\frac{n}{2}$ edges incident with $v$ if the corresponding facets in $K$ and two vertices $u_1$ and $u_2$ of $M$ are ends of an edge of $M$ if the corresponding facets in $K$ have an edge in common. The well-known maps of type $\{3, 6\}$ and $\{6, 3\}$ on the surface of torus are examples of mutually dual maps.

Consider a polyhedral map $K$ on a surface $S$ that has $n$ vertices.

**Definition 1.1.** (See Maity and Upadhyay [7]) Let $M$ denote the dual map of $K$. Let $T := (V, E)$ denote a tree in the edge graph $EG(M)$ of $M$. We say that $T$ is a proper tree if the following conditions hold:

1. $\sum_{i=1}^{k} \deg(v_i) = n + 2(k - 1)$, where $V = \{v_1, v_2, ..., v_k\}$ and $\deg(v)$ denotes degree of $v$ in $EG(M)$,
2. whenever two vertices $u_1$ and $u_2$ of $T$ lie on a face $F$ in $M$, a path $P[u_1, u_2]$ joining $u_1$ and $u_2$ in the boundary $\partial F$ of $F$ is a subtree of $T$, and
3. any path $P$ in $T$ which lies in a face $F$ of $M$ is of length at most $q - 2$, where $q$ is the length of $\partial F$. 

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Definition 1.2. Let $M$ denote the dual map of a polyhedral map $K$ on $n$ vertices. Let $H := (V, E)$ denote a subgraph in the edge graph $\text{EG}(M)$ of $M$. We say that $H$ is an admissible graph if the following conditions hold:

1. $H$ has a decomposition into proper trees $T_1, T_2, \ldots, T_r$ such that $H = T_1 \cup T_2 \cup \ldots \cup T_r$ and $T_i \neq T_j$ for $i \neq j, i, j \in \{1, \ldots, r\},$
2. $T_i \cap T_j$ is a set of paths and for $v \in V(T_i \cap T_j)$ we have $\deg(v)$ in $\text{EG}(M)$ is equal to $\deg(v)$ in $T_i \cup T_j$, for $i \neq j$ and $i, j \in \{1, \ldots, r\}$, and
3. the graph $T_i \cup T_j$ does not contain a pair of vertices $u_i, u_j$ with $u_i \in V(T_i)$ and $u_j \in V(T_j)$ such that $u_i u_j \in E(\text{EG}(M))$ and $u_i u_j \notin E(T_i \cup T_j)$ for $i \neq j$ and $i, j \in \{1, \ldots, r\}$. 

Remark 1.1.: Let $v \in V(T_{i_1}), \ldots, V(T_{i_t}), i_1, \ldots, i_t \in \{1, \ldots, r\}$, then $\sum_{v \in V(H)} t \deg(v) = rn + 2 \sum_{i=1}^{r} (k_i - 1)$, where $H = T_1 \cup T_2 \cup \ldots \cup T_r$, $n = |V(\text{EG}(K))|$ and $k_i = |V(T_i)|$.

By the Definition 1.1 we have $\sum_{j=1}^{k_i} \deg(v_j) = n + 2(k_i - 1)$ for the proper tree $T_i$ and $1 \leq i \leq r$, hence $\sum_{i=1}^{r} \sum_{j=1}^{k_i} \deg(v_j) = \sum_{i=1}^{r} (n + 2(k_i - 1)) = rn + 2 \sum_{i=1}^{r} (k_i - 1)$.

Proposition 1.1. [Maity, Upadhyay] [7] The edge graph $\text{EG}(K)$ of a map $K$ on a surface has a contractible Hamiltonian cycle if and only if the edge graph of the corresponding dual map of $K$ has a proper tree.

The main result of this note is:

Theorem 1.1. Let $K$ be a map on the surface $S$ with $n$ vertices. Then, $K$ contains $r$ edge-disjoint contractible Hamiltonian cycles, if and only if the dual map $M$ of $K$ contains an admissible graph $H$ that has a decomposition into $r$ proper trees.

In particular, we prove:

Corollary 1.1. Let $K$ be a map on the surface $S$ with $n$ vertices. Then, $K$ contains $r$ face-disjoint contractible Hamiltonian cycles, if and only if the dual map $M$ of $K$ contains an admissible graph $H$ that has a decomposition into $r$ disjoint proper trees.

In the next section, we give examples of an admissible graph and the existence of edge- and face-disjoint contractible Hamiltonian cycles in polyhedral maps. Then, in the following section we present the proofs of Theorem 1.1 and Corollary 1.1.
2. Examples

Example 2.1. Figure 1 depicts a triangulation of a surface $M_1$ of $\chi = 0$ on 7 vertices (see Datta and Upadhyay [3]). $K$ depicts the dual of $M_1$ in Figure 2. Graph $H := (V, E)$ where $V := \{w_1, w_2, w_4, w_6, w_9, w_{10}, w_{13}, w_{14}\}$ and $E := \{w_1w_2, w_1w_6, w_1w_{14}, w_{13}w_{14}, w_4w_{13}, w_9w_{14}, w_9w_{10}\}$ is an admissible graph in $K$. Let $T_1 := (V_1, E_1)$ where $V_1 := \{w_1, w_2, w_9, w_{10}, w_{14}\}$ and $E_1 := \{w_1w_2, w_1w_{14}, w_9w_{14}, w_9w_{10}\}$, and $T_2 := (V_2, E_2)$ where $V_2 := \{w_1, w_4, w_6, w_{13}, w_{14}\}$ and $E_1 := \{w_1w_6, w_1w_{14}, w_{13}w_{14}, w_4w_{13}\}$. Then, graph $H$ has a decomposition into $T_1$ and $T_2$.

Example 2.2. Figure 3 depicts a triangulation of a surface $M_2$ of $\chi = -3$ on 9 vertices taken from Lutz [6]. $\partial D_1 = C(1, 6, 4, 2, 3, 5, 7, 9, 8)$ in Figure 4 and $\partial D_2 = C(5, 2, 7, 1, 3, 8, 6, 9, 4)$ in Figure 5 depict edge-disjoint contractible Hamiltonian cycles in $M_2$. 

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Example 2.3. Figure 6 Lutz [6] depicts a triangulation of a surface $M_3$ of $\chi = -10$ on 12 vertices. This triangulation contains two face disjoint cycles $\partial D'_1 = C(1, 7, 8, c, 4, 5, 3, 9, 6, a, b, 2)$ in Figure 7 and $\partial D'_2 = C(1, 8, 2, 5, c, 6, 7, b, 3, a, 9, 4)$ in Figure 8 as shown below.

![Figure 6: $M_3$](image)

![Figure 7: $D'_1$](image)

![Figure 8: $D'_2$](image)

3. Proof of the Theorem 1.1

**Proof of Theorem 1.1**: Let $M$ be as in the statement of Theorem 1.1 and containing an admissible graph $H$. By Definition 1.2, it has a decomposition $H = T_1 \cup T_2 \cup \cdots \cup T_r$. Then by the Proposition 1.1 see Maity and Upadhyay [7], the map $K$ contains contractible Hamiltonian cycles $C_i$ corresponding to $T_i$ for $1 \leq i \leq r$. Hence the map $K$ contains $r$ contractible Hamiltonian cycles. We now show that these cycles are pairwise edge-disjoint.

Suppose, on the contrary, $E(C_i) \cap E(C_j)$ contains an edge $uv$. Then $uv$ belongs to two faces, say, $F_1$ and $F_2$. Let $D(C_i)$ denote the 2-disk which is bounded by the cycle $C_i$ and $v_{F_1}$ denote the vertex corresponding to $F_1$ in the dual. Two situations may arise. In the first, if $F_1 \in DC_i$ and $F_2 \in DC_j$, then edge $v_{F_1}v_{F_2}$ does not belong to the graph $T_i \cup T_j$. That is, $v_{F_1}v_{F_2} \notin E(T_i \cup T_j)$ and $v_{F_1}v_{F_2} \notin E(EG(M))$. This contradicts the condition 3 of Definition 1.2. Further, in the second situation if one of the two faces $F_1$ and $F_2$, say $F_1$, belongs to both disks $DC_i$ and $DC_j$ then $F_1$ lies in both disks. Hence the degree of $v_{F_1}$ in $T_i \cup T_j$ is less than the degree of $v_{F_1}$ in $EG(M)$. This contradicts the condition 2 in Definition 1.2. Therefore $E(C_i) \cap E(C_j) = \emptyset$ for $i \neq j$ and $i, j \in \{1, \ldots, r\}$. Hence the map $K$ contains $r$ edge-disjoint contractible Hamiltonian cycles.

Suppose the map $M$ has $r$ edge-disjoint Hamiltonian cycles $C_1, C_2, \ldots, C_r$ and let the dual of the disk $DC_i$ be the tree $T_i$. We define $H := T_1 \cup T_2 \cup \cdots \cup T_r$. Since all the $T_i$s are distinct proper trees, it is easy to check that $H$ satisfies the condition 1 in Definition 1.2. Suppose there are two trees $T_i$ and $T_j$ such that the graph $T_i \cap T_j$ contains a vertex $v$ with $\deg(v)$ in the graph $EG(M)$ that is greater than its degree in $T_i \cup T_j$. Thus there exists an edge $vw$ that does not belong to the graph $T_i \cup T_j$. Consider the dual face $F_v$ corresponding to vertex $v$. Face $F_v$ belongs
to both disks $DC_i$ and $DC_j$ as $v$ belongs to $V(T_i \cap T_j)$. So the dual edge corresponding to $vw$ shall lie in the boundary of the 2-disks $DC_i$ and $DC_j$. Hence $C_i$ and $C_j$ are not edge-disjoint. This is a contradiction. Hence $\deg(v) \in EG(M)$ is greater than $\deg(v)$ in $T_i \cup T_j$ for all the vertices of $T_i \cap T_j$. This gives the condition 2 in the Definition 1.2. Let $u_i \in V(T_i)$ and $u_j \in V(T_j)$ be such that $u_iu_j \in E(EG(M))$ and $u_iu_j \notin E(T_i \cup T_j)$. Then face $F_{u_i}$ belongs to the disk $DC_i$ and face $F_{u_j}$ belongs to the disk $DC_j$. Moreover, the dual edge corresponding to $u_iu_j$ will lie in both faces $F_{u_i}$ and $F_{u_j}$. Hence edge $u_iu_j$ will be on the boundary of both the 2-disks $DC_i$ and $DC_j$. Therefore both the cycles $C_i$ and $C_j$ contain the dual edge corresponding to $u_iu_j$. So $C_i$ and $C_j$ are not edge-disjoint. This is a contradiction. So we see that the condition 3 in Definition 1.2 is also satisfied. Thus $H$ is the required admissible graph.

**Proof of Corollary 1.1:** To prove the corollary we proceed exactly same as in the previous proof of Theorem 1.1 and we use disjoint proper trees instead of proper trees.

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