On classes of neighborhood resolving sets of a graph

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Abstract

Let \( G = (V, E) \) be a simple connected graph. A subset \( S \) of \( V \) is called a neighbourhood set of \( G \) if \( G = \bigcup_{v \in S} N[v] \), where \( N[v] \) denotes the closed neighbourhood of the vertex \( v \) in \( G \). Further for each ordered subset \( S = \{s_1, s_2, \ldots, s_k\} \) of \( V \) and a vertex \( u \in V \), we associate a vector \( \Gamma(u/S) = (d(u, s_1), d(u, s_2), \ldots, d(u, s_k)) \) with respect to \( S \), where \( d(u, v) \) denote the distance between \( u \) and \( v \) in \( G \). A subset \( S \) is said to be resolving set of \( G \) if \( \Gamma(u/S) \neq \Gamma(v/S) \) for all \( u, v \in V - S \). A neighbouring set of \( G \) which is also a resolving set for \( G \) is called a neighbourhood resolving set (\( nr \)-set). The purpose of this paper is to introduce various types of \( nr \)-sets and compute minimum cardinality of each set, in possible cases, particularly for paths and cycles.

Keywords: resolving set, neighbourhood set, neighbourhood resolving sets.

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1. Introduction

All the graphs considered in this paper are connected, simple, undirected, and finite. Let \( p_1 \) be a graph property satisfied by at least one subset of vertices of \( G \). Then such subsets \( S \) which satisfies the property \( p_1 \) are called \( p_1 \)-sets of \( G \). A \( p_1 \)-set \( S \) of \( G \) is called a \( P_1 \)-set if \( \bar{S} \) is not a \( p_1 \)-set of \( G \). A \( p_1^* \)-set of \( G \) is a set \( S \) such that both \( S \) and \( \bar{S} \) are \( p_1 \)-sets of \( G \). A \( P_1^* \)-set of \( G \) is a...
set $S$ such that both $S$ and $\overline{S}$ are not $p_1$-sets of $G$. If $p_2$ is another graph property satisfied by any subset of vertices of $G$, then a set $S$ which satisfies both the property $p_1$ and $p_2$ is called a $p_1p_2$-set. If $S$ is a $p_1$-set and also a $p_2^*$-set, then we say $S$ is a $p_1p_2^*$-set. Similarly, $p_1p_2p_3$-sets, $p_1P_2p_3$-sets, $p_1P_2P_3^*$-sets, etc., are defined.

A pq-set is said to be a minimal pq-set of $G$ if none of its proper subsets are pq-set of $G$. The minimum cardinality of a minimal pq-set of $G$ is called lower pq number of $G$ and is denoted by $l_{pq}(G)$.

Let $G$ be a graph and $v$ be a vertex of $G$. Let $N(v)$ be the set of vertices adjacent to $v$ in $G$ and $N[v] = N(v) \cup \{v\}$. A subset $S$ of vertex set of $G$ is called a neighbourhood set or an $n$-set of $G$ if $G = \bigcup_{v \in S} \langle N[v] \rangle$, where $\langle N(v) \rangle$ is the subgraph of $G$ induced by the set $S$. Further a subset $S$ of a vertex set of $G$ is called a resolving set or an $r$-set of $G$ if for each pair $u, v \notin S$ there is a vertex $w \in S$ with the property that $d(v, w) \neq d(u, w)$.

The metric dimension of $G$, denoted by $\beta(G)$, is the minimum cardinality of all the resolving sets of $G$. A resolving set with minimum cardinality is called a metric basis. The concept of Metric dimension was introduced by F. Harary and R.A. Melter [3] and independently by P.J. Slater [13] under the term locating set. For more works on metric dimension, we refer [2, 5, 6, 7, 10, 11, 12, 14, 15].

The neighbourhood number of a graph was introduced by E. Sampathkumar et al. in [8] and studied the relationship of $l_n(G)$ (denoted by $n_0$) with some other known graph parameters.

If $S$ is both neighbourhood and resolving, then in the above notation we write $S$ as an $nr$-set. The terms not defined here may found in [1]. Throughout this paper $P_k$ denotes a path on $k$ vertices with a vertex set $V = \{v_i : 1 \leq i \leq k\}$ and an edge set $E = \{v_iv_{i+1} : 1 \leq i \leq k-1\}$. Similarly, $C_k$ denotes a cycle on $k$ vertices with a vertex set $V = \{v_i : 1 \leq i \leq k\}$ and an edge set $E = \{v_iv_{i+1}\} \cup \{v_1v_k\}$.

**Remark 1.1.** From the definition of a resolving set, it is clear that any 2-element subset of vertices of a path $P_k$ is always an $r$-set of $P_k$. In fact, if $S = \{a, b\}$ and $u, v$ be arbitrary vertices of $P_k$ such that $d(u, a) = d(v, a)$, then $a$ is the central vertex of the $uv$-path in $P_k$, but then exactly one of the paths, $ub$-path or $vb$-path, in $P_k$ contains the vertex $a$ and hence $d(u, b) \neq d(v, b)$.

**Remark 1.2.** A singleton set $S = \{v\}$ is a resolving set of a path $P$ if and only if $v$ is an end vertex of $P_k$.

**Remark 1.3.** A subset of vertices of $P_k$ containing an end vertex is always a resolving set of $P_k$.

**Remark 1.4.** For a connected graph $G$ of order $k$, every subset of cardinality at least $k - 1$ is always an $n$-set.

**Remark 1.5.** Since a superset of any $r$-set of a graph $G$ is also an $r$-set of the graph $G$, it follows from Remark 1.1 that every $i$-element subset of the vertex set of a path $P_k$ is always an $r$-set of $P_k$, for every $i, 2 \leq i \leq k$.

**Observation 1.1.** Every $n$-set of a path $P_k$ has at least 2 elements, whenever $k \geq 4$.

**Observation 1.2.** Every $r$-set of a path $P_k$, $2 \leq k \leq 3$, contains a pendent vertex.

We recall the following for immediate reference;
**Theorem 1.1** (S. Khuller, B. Raghavachari, and A. Rosenfeld [6]). For a simple connected graph \( G \), \( \beta(G) = 1 \) if and only if \( G \cong P_k \).

**Theorem 1.2** (F. Harary and R.A. Melter [3]). For any integer \( k \geq 3 \), the metric dimension of a cycle on \( k \) vertices is 2.

**Theorem 1.3** (B. Sooryanarayana [14]). A graph \( G \) with \( \beta(G) = k \), cannot contain \( k_{2k+1} - (2^{k-1} - 1)e \) as a subgraph.

**Theorem 1.4** (E. Sampathkumar and Prabha S. Neeralagi [9]). For a path \( P_k \) on \( k \) vertices, the lower neighbourhood number \( l_n(P_k) = \lceil \frac{k}{2} \rceil \).

**Theorem 1.5** (E. Sampathkumar and Prabha S. Neeralagi [8]). For a cycle \( C_k \) of length \( k \geq 4 \), the lower neighbourhood number \( l_n(C_k) = \lfloor \frac{k}{2} \rfloor \).

**Theorem 1.6** (E. Sampathkumar and Prabha S. Neeralagi [8]). A set \( S \) of vertices of a graph \( G \) is an \( n \)-set if and only if every line of \( (V(G) - S) \) belongs to a triangle one of whose vertices belong to \( S \).

### 2. \( nr \)-sets and Dimensions of a Path

**Theorem 2.1.** For any integer \( k \geq 1 \), \( l_n(P_k) = \begin{cases} \lceil \frac{k}{2} \rceil, & \text{for } k \leq 3, \\ \lceil \frac{k}{2} \rceil, & \text{for } k \geq 4. \end{cases} \)

**Proof.** For the case \( k = 1, 2 \), it is easy to see that any singleton subset of \( V(P_k) \) is always an \( nr \)-set. For \( k = 3 \), a singleton subset containing an end vertex is not an \( n \)-set and a singleton subset containing the central vertex is not an \( r \)-set of \( P_3 \). Therefore, every \( nr \)-set should have at least two elements. Further, as any subset \( S \subseteq V(P_3) \) with \( |S| = 2 \) is an \( nr \)-set for \( P_3, l_{nr}(P_3) = 2 \). Now for \( k \geq 4 \), any subset \( S \subseteq V(P_k) \) containing two or more elements is always an \( r \)-set (by Remark 1.5). Therefore, as \( l_n(P_k) \geq 2 \) for all \( k \geq 4 \), it follows that \( l_{nr}(P_k) = l_n(P_k) = \lceil \frac{k}{2} \rceil \) (by Theorem 1.4). \( \square \)

**Theorem 2.2.** For any integer \( k \geq 1 \), \( l_{nr}(P_k) = \begin{cases} k, & \text{for } k = 1, 2, \\ k - 1, & \text{for } k \geq 3. \end{cases} \)

**Proof.** Let \( S \) be an \( nR \)-set of a path \( P_k \). Then \( S \) is an \( r \)-set and \( \bar{S} \) is not an \( r \)-set. So, by Remark 1.1 and Remark 1.3, it follows that a minimal \( R \)-set \( S \) should contain both the end vertices and is of cardinality at least \( k - 1 \) whenever \( k \geq 3 \) or exactly \( k \) if \( k \leq 2 \). But then, by Remark 1.4, \( S \) is an \( n \)-set of \( P_k \). Hence \( l_{nr} = k - 1 \) if \( k \geq 3 \) or \( l_{nr} = k \) if \( k \leq 2 \). \( \square \)

**Theorem 2.3.** For any integer \( k \geq 1 \), \( l_{NR}(P_k) = \begin{cases} k, & \text{for } k \leq 2, \\ k - 1, & \text{for } k \geq 3. \end{cases} \)

**Proof.** Follows by the proof of the previous Theorem 2.2, as each \( nR \)-set \( S \) of \( P_k \) is also an \( NR \)-set of \( P_k \) (Since the set \( \bar{S} \) contains at most one element which is non-end vertex and hence by Observation 1.1 and Observation 1.2, \( \bar{S} \) is not an \( n \)-set if \( k \neq 3 \) and not an \( r \)-set if \( k = 3 \)). \( \square \)
Lemma 2.1. Any independent set $S$ of vertices of a path $P_k$ contains more than $\frac{k}{2}$ vertices is always an $n$-set.

Proof. Let $S$ be an independent set of the path $P_k$ contains more than $\frac{k}{2}$ vertices. Then $k$ is odd, $S = \{v_1, v_3, v_5, \ldots, v_{k-2}, v_k\}$, and $\bigcup_{v \in S} N[v] = V(P_k)$. Let $e_i = v_i v_{i+1}$ be an edge of $P_k$, $1 \leq i \leq k-1$. Then $e_i$ is an edge of either $\langle N[v_i]\rangle$ or $\langle N[v_{i+1}]\rangle$ depending upon whether $i$ is odd or even. Hence for each $i$, the edge $e_i \in \langle N[v_j]\rangle$ for some odd $j$. Therefore, $\bigcup_{v \in S} \langle N[v]\rangle = G$. \hfill \blacksquare

Similarly, we prove:

Lemma 2.2. Any independent set $S$ of vertices of a path $P_{2k}$ contain (at least) $k$ vertices is always an $n$-set of $P_{2k}$.

Lemma 2.3. If $S$ is an $n$-set of the graph $G$, then $\bar{S}$ is independent.

Proof. If not, suppose that $\bar{S}$ contains two adjacent vertices say $x$ and $y$, then the edge $xy$ is not in the graph $\bigcup_{v \in S} \langle N[v]\rangle = G$, a contradiction to the fact that $S$ is an $n$-set. \hfill \blacksquare

Theorem 2.4. For any integer, $l_{N_r}(P_k) = \begin{cases} k, & \text{for } k = 1, 2; \\ \lceil \frac{k}{2} \rceil, & \text{for } k \geq 3. \end{cases}$

Proof. The result is obvious for $k \leq 4$. Consider the case $k \geq 5$, let $S$ be an $N$-set of $P_k$. Then $S$ is an $n$-set, so by Theorem 1.4, $|S| \geq \lceil \frac{k}{2} \rceil \geq 2$ vertices and hence by Remark 1.5, $S$ is also an $r$-set. If $k$ is odd and $|S| = \lfloor \frac{k}{2} \rfloor$, then $|\bar{S}| \geq \lfloor \frac{k}{2} \rfloor$, so by Lemma 2.3 and Lemma 2.1 the subset $\bar{S}$ is an $n$-set, a contradiction to the fact that $\bar{S}$ is an $N$-set. Therefore, $|S| \geq \lceil \frac{k}{2} \rceil$ for all $k$ implies that $l_{N_r}(P_k) \geq \lceil \frac{k}{2} \rceil$. On the other hand, it is easy to see that the set $S = \{v_{2[l/2]}, v_{2[l/2]-2}, \ldots, v_2\} \cup \{v_p\} \cup \{v_{[k/2]+1}, v_{[k/2]+3}, \ldots, v_{k-1}\}$ is an $Nr$-set of $P_k$ with $|S| = \lceil \frac{k}{2} \rceil$ where $p = 2$, if $k$ is even and $p = 1$, if $k$ is odd. Thus, $l_{N_r}(P_k) \leq \lceil \frac{k}{2} \rceil$. \hfill \blacksquare

Theorem 2.5. For any positive integer $k$, $k \neq 1, 3$, $l_{n^*r}(P_k) = l_{n^*r}(P_k) = l_{n^*r^*}(P_k) = \lceil \frac{k}{2} \rceil$.

Proof. The result is obvious for $k = 2$. Now for the case $k \geq 4$, as every $n^*$-set is an $n$-set, we have $|S| \geq \lceil \frac{k}{2} \rceil$ (by Theorem 1.4) and hence $l_{n^*r^*}(P_k)$, $l_{n^*r^*}(P_k)$, $l_{n^*r^*}(P_k) \geq \lceil \frac{k}{2} \rceil$. On the other hand, we see that the set $S = \{v_2, v_4, \ldots, v_{2[\frac{k}{2}]}\}$ is an $n$-set of $P_k$. So, by Lemma 2.1 or Lemma 2.2 respectively when $k$ is odd or even, the set $\bar{S}$ is an $n$-set. Since $k \geq 4$, both $S$ and $\bar{S}$ have at least two elements and hence each of them will resolve $P_k$. Hence $S$ is an $n^r$-set as well as $n^r^*$-set and $n^r^*$-set with $|S| = \lceil \frac{k}{2} \rceil$. Therefore, $l_{n^*r^*}(P_k) \leq \lceil \frac{k}{2} \rceil$, $l_{n^*r^*}(P_k) \leq \lceil \frac{k}{2} \rceil$, and $l_{n^*r^*}(P_k) \leq \lceil \frac{k}{2} \rceil$. \hfill \blacksquare

Remark 2.1. When $k = 1$, $\bar{S}$ is empty. Hence $n^*$-set as well as $r^*$-set are not defined. But when $k = 3$, it is easy to see that $l_{n^*r}(P_3) = l_{n^*r^*}(P_3) = 2$. However, $P_3$ has no $n^r^*$-set $S$ and hence $l_{n^*r^*}(P_3)$ is not defined.

Theorem 2.6. For integer $k \geq 4$, $l_{N^*r}(P_k) = l_{N^*r^*}(P_k) = 2$. 

Proof. Let $S$ be an $N^*r$-set of $P_k$. Then $S$ is not an $n$-set, $\bar{S}$ is not an $n$-sets, and $S$ is an $r$-set. Now, if $|S| = 1$, then $S$ contains only an end vertex of $P_k$ (by Remark 1.2) and hence $|\bar{S}| = k - 1$. But then, $\bar{S}$ is an $n$-set (by Remark 1.4), a contradiction. Thus, $2 \leq |S| \leq k - 2$. Hence $l_{N^*r}(P_k) \geq 2$ and $l_{N^*r}(P_k) \geq 2$. On the other hand, take $S' = \{v_1, v_2\}$. The set $S'$ as well as $\bar{S}'$ are not $n$-sets (since the edge $v_1v_2$ is not an edge of $\bigcup_{v \in S'}(N[v])$). But $S'$ is an $r$-set (and $\bar{S}'$ is also an $r$-set), whenever $k \geq 4$ (since $|S'| = 2$ and $|\bar{S}'| \geq 2$ and by Remark 1.5). Hence $l_{N^*r}(P_k) \leq 2$ and $l_{N^*r}(P_k) \leq 2$. \hfill $\square$

Remark 2.2. If $k \leq 3$, for every subset $S$ of $V(P_k)$, either $S$ or $\bar{S}$ is an $n$-set. Hence no $N^*$-set exists.

We end up this section with the following theorem, whose proof follows similar to the proof of Theorem 2.4.

Theorem 2.7. For any integer $k \geq 3$, $l_{N^*r}(P_k) = \lceil \frac{k}{2} \rceil$.

When $k = 1$, no $r^*$-set exists and when $k = 2$, no $N$-set exists. It is easy to see that the other sets like $nR^*$-set, $n^*R^*$-set, $N^*R^*$-set, and $N^*R^*$-set are not exists in any path due to the non-existence of $R^*$-sets. Finally, the non-existence of $N^*R^*$-set is due to the fact that if $S$ is any such set, then its complement should contains exactly one vertex other than the end vertex to become an $R^*$-set implies that the set $S$ is an $n$-set (so not an $N^*$-set).

3. $nr$-sets and Dimensions of a Cycle

We first restate the consequences of Theorem 1.6 as;

Lemma 3.1. Let $e = xy$ be an edge of a graph $G$ such that $e$ is not an edge of a triangle in $G$ and $S$ be an $n$-set of $G$. Then $x, y \in N[v]$ for some $v \in S$ if and only if $x = v$ or $y = v$.

Lemma 3.2. If $S$ is an $n$-set of a graph $G$, then for each edge $e = xy$ there exists a vertex $v$ in $S$ such that both $x, y \in N[v]$.

Theorem 3.1. For each integer $i \geq 3$, every $i$-element subset $S$ of vertices of a cycle $C_k$ is always an $r$-set.

Proof. Let $S$ be a subset of the vertices of $C_k$ with cardinality at least 3. Let $a, b, c \in S$ and $x, y$ be any two vertices of cycle $C_k$ for $k \geq 3$. If possible, let $d(a, x) = d(a, y)$ and $d(b, x) = d(b, y)$. Then $a$ and $b$ lie in distinct $xy$-paths in $C_k$ and $C_k$ is an even cycle. In case if $c$ lies between $a$ and $x$, then $d(c, x) < d(c, y)$ and hence $c$ resolves the pair $x, y$. Similarly, other cases follows by symmetry. \hfill $\square$

Remark 3.1. A set containing two adjacent vertices of a cycle $C_k$ is always an $r$-set of $C_k$ for each $k \geq 3$.

Theorem 3.2. For any integer $k \geq 3$, $l_{nr}(C_k) = \begin{cases} 3, & \text{for } k = 4, \\ \lceil \frac{k}{2} \rceil, & \text{otherwise} \end{cases}$
Proof. In the case $k = 4$, it follows by Theorem 1.4 that $|S| \geq 2$. If $|S| = 2$, then $S$ contains two adjacent vertices (else it is not an $r$-set). But then, $\langle V(C_4) - S \rangle$ contains an edge and hence by Theorem 1.6, $C_k$ should contain a triangle, a contradiction. Hence every $nr$-set should have at least 3 elements. For the case $k \geq 5$, it is easy to see from Theorem 1.5 and Theorem 1.6 that the set $S = \{v_1, v_3, v_5, \ldots, v_{2^{\lceil \frac{k}{2} \rceil} - 1}\}$ is an $n$-set and hence by Theorem 3.1, it follows that $l_{nr}(C_k) = |S| = \lceil \frac{k}{2} \rceil$.

\[\blackbox{\textbf{Theorem 3.3.} For any integer } k \geq 4, l_{N^r_\bullet}(C_k) = l_{N^r_\bullet^*}(C_k) = 2\]

Proof. Let $e = xy$ be an edge of $C_k$ and $S = \{x, y\}$. Then $S$ is a resolving set for $C_k$. Now as $k \geq 4$, there is an edge $e_1 = uv$ not adjacent to $e$. So, by Lemma 3.2, $S$ is not an $n$-set (Since $C_k$ has no triangle and $u, v \notin S$). Hence $S$ is an $N^r_\bullet$-set. Further as $\beta(C_k) = 2$, there are no singleton $r$-sets implies that the above set $S$ is a minimal $N^r_\bullet$-set, $l_{N^r_\bullet}(C_k) = 2$. Also, $\bar{S}$ contains at least 3 vertices if $k > 4$ and 2 adjacent vertices if $k = 4$. So, by Theorem 3.1 and Remark 3.1, $\bar{S}$ is an $r$-set. Therefore, $S$ is also an $N^r_\bullet^*$-set of minimum cardinality, so $l_{N^r_\bullet^*}(C_k) = 2$ for all $k \geq 4$.

\[\blackbox{\textbf{Lemma 3.3.} Let $S$ be a minimal $n$-set of a graph $G$ with $\Delta(G) = 2$ and $H = \langle S \rangle$. Then $\Delta(H) < 2$.}

Proof. If possible, let $S$ be a minimal $n$-set of $G$ and $\Delta(H) = 2$. Then there exists $a, b, c \in S$, Such that $ab, bc \in E(G)$. Consider the set $S' = S - \{b\}$. Since $\Delta(G) = 2$, we have $\deg_G(b) = 2$ and hence $b$ is adjacent to only $a$ and $c$. Therefore, $S'$ covers all the edges of $G$ incident with $b$ as well as other edges of $G$ (Since other edges covered by $S$). This shows that $S'$ is an $n$-set, a contradiction to the minimality of $S$.

\[\blackbox{\textbf{Theorem 3.4.} For any integer } k > 4, l_{N^r}(C_k) = l_{N^r^*}(C_k) = \lceil \frac{k+1}{2} \rceil. \text{ Also, } l_{N^r}(C_4) = 3.\]

Proof. Let $S$ be a minimal $Nr$-set of cycle $C_k$, $k > 4$. Then $S$ is an $n$-set, therefore by Theorem 1.5, $|S| \geq \lceil \frac{k}{2} \rceil$ and by Lemma 3.3 the induced subgraph $\langle S \rangle$ has no two adjacent edges of $G$ (i.e $\deg_{\langle S \rangle}(v) \leq 1, \forall v \in S$). So, if $k$ is even and $|S| = \lceil \frac{k}{2} \rceil$, then in the view of Lemma 3.2, we have, $\bar{S}$ is an $n$-set, a contradiction to the fact that $S$ is an $N$-set. Thus, $|S| \geq \lceil \frac{k+1}{2} \rceil$ implies that $l_{N^r}(C_k) \geq \lceil \frac{k+1}{2} \rceil$ and $l_{N^r^*}(C_k) \geq \lceil \frac{k+1}{2} \rceil$. On the other hand, consider the set $S = \{v_1, v_3, v_5, \ldots, v_{2^{\lceil \frac{k}{2} \rceil} - 1}\} \cup \{v_{k-1}\}$. The set $S$ is an $n$-set with $|S| = \lceil \frac{k+1}{2} \rceil$ and $\bar{S} = \lceil \frac{k-1}{2} \rceil < \lceil \frac{k}{2} \rceil$ and hence $\bar{S}$ is not an $n$-set implies that $S$ is an $N$-set. Finally, as $k > 4$, we have $|S| > 3$. Hence by Theorem 3.1, $S$ is also an $r$-set. Thus, $l_{N^r}(C_k) \leq \lceil \frac{k+1}{2} \rceil$. Further when $k = 5$, it is easy to see that $\bar{S}$ contains an adjacent pair of vertices and when $k > 5$, the set $\bar{S}$ has at least 3 vertices. Hence by Remark 3.1 and the 3.1, the set $S$ is also an $r^*$-set. Hence it also follows that $l_{N^r^*}(C_k) \leq \lceil \frac{k+1}{2} \rceil$. Lastly, the case $k = 4$ follows easily.

\[\blackbox{\textbf{Remark 3.2.} When $k = 3$, it is easy to see that for every $nr$-set $S$ of $C_3$, the set $\bar{S}$ is also an $n$-set and no $N$-set exists.}

\[\blackbox{\textbf{Theorem 3.5.} For any integer } k > 4, l_{nr^*}(C_k) = \lceil \frac{k}{2} \rceil\]

Proof. Follows immediately by Theorem 1.4 and Theorem 3.1, as $l_{nr^*}(C_k) = l_n(C_k) = \lceil \frac{k}{2} \rceil$ for all $k > 4$.\]

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Remark 3.3. Since $\beta(C_k) = 2$, every $r$-set of $C_k$ should have at least 2 elements. Therefore, for the existence of an $r^*$-set of a cycle $C_k$, $k$ should be at least 5. Further when $k = 3$ or $4$, it is easy to see that for every $nr$-set $S$ of $C_k$ we get $|S| = 1$, and hence $S$ is not an $r^*$-set.

**Theorem 3.6.** For any integer $k \geq 4$, $l_{NR}(C_k) = l_{nR}(C_k) = \begin{cases} k - 2, & \text{when } k \text{ is even and } k \neq 4, \\ k - 1, & \text{otherwise}. \end{cases}$

**Proof.** Since $\beta(C_k) = 2$, any two vertices of $C_k$ resolves $C_k$ except the case $k$ is even and the vertices are diagonally opposite. Therefore, for $k > 4$, every $R$-set $S$ should have minimum of $k - 1$ vertices whenever $k$ is odd and $k - 2$ if $k$ is even. In either of the cases, the subgraph $\bigcup_{v \in S} N[v] \cong C_k$ for every $R$-set $S$ and $\bigcup_{v \in S} N[v] \neq C_k$ for $k \neq 4$ and hence $S$ is an $n$-set as well as an $N$-set. When $k=4$, every $N$-set should have at least 3 elements and such a set $S$ with $|S| = 3$ is always an $R$-set. \hfill \Box

**Theorem 3.7.** For every integer $k \geq 3$, $l_{n^*r^*}(C_{2k}) = l_{n^*r}(C_{2k}) = k$.

**Proof.** Let $S$ be an $n^*$-set. Then $S$ and $\overline{S}$ both are edge covering of $C_{2k}$. Since edge covering number of $C_{2k}$ is $k$, $|S| = |\overline{S}| = k$. Also, both $S$ and $\overline{S}$ are $r$-sets (since $k \geq 3$). Finally, every maximal independent set $S$ is an $n^*r^*$-set as well as $n^*r$-set. Hence the result. \hfill \Box

Remark 3.4. For an odd cycle, no $n^*$-set exists as each $n$-set contains both end vertices of an edge (so $\overline{S}$ is not an $n$-set, by Lemma 3.2).

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