Characterizing all trees with locating-chromatic number 3

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Abstract

Let $c$ be a proper $k$-coloring of a connected graph $G$. Let $\Pi = \{S_1, S_2, \ldots, S_k\}$ be the induced partition of $V(G)$ by $c$, where $S_i$ is the partition class having all vertices with color $i$. The color code $c_\Pi(v)$ of vertex $v$ is the ordered $k$-tuple $(d(v, S_1), d(v, S_2), \ldots, d(v, S_k))$, where $d(v, S_i) = \min\{d(v, x) | x \in S_i\}$, for $1 \leq i \leq k$. If all vertices of $G$ have distinct color codes, then $c$ is called a locating-coloring of $G$. The locating-chromatic number of $G$, denoted by $\chi_L(G)$, is the smallest $k$ such that $G$ possess a locating $k$-coloring. Clearly, any graph of order $n \geq 2$ has locating-chromatic number $k$, where $2 \leq k \leq n$. Characterizing all graphs with a certain locating-chromatic number is a difficult problem. Up to now, all graphs of order $n$ with locating chromatic number $2$, $n - 1$, or $n$ have been characterized. In this paper, we characterize all trees whose locating-chromatic number is 3. We also give a family of trees with locating-chromatic number 4.

Keywords: Locating-chromatic number, graph, tree.
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1. Introduction

Chartrand et al. [8] initiated the study on the locating-chromatic number of a graph. This notion is a special case of the partition dimension of a graph, namely the smallest integer $k$ in which there exists a $k$-partition $\Pi$ of the graph such that the coordinates of all vertices with respect to $\Pi$ are distinct. Since then, various results have been obtained by different authors. However,
determining the locating-chromatic number of any graph in general is classified as an NP-hard problem [8]. Furthermore, characterizing all graphs with a certain locating-chromatic number is also a difficult question.

In this paper, we consider only simple connected graphs. Let \( G(V, E) \) be a graph. The distance \( d(u, v) \) from vertex \( u \) to vertex \( v \) in \( G \) is the length of a shortest path from \( u \) to \( v \). For \( S \subseteq V(G) \), define the distance \( d(v, S) \) from vertex \( v \) to set \( S \) as \( \min\{d(v, x) | x \in S\} \). Let \( c \) be a \( k \)-coloring of \( G \) and \( \Pi = \{S_1, S_2, \ldots, S_k\} \) be a partition of \( V(G) \) induced by \( c \), where \( S_i \) is the set of vertices receiving color \( i \). The color code \( c_{\Pi}(v) \) of \( v \) is defined as the ordered \( k \)-tuple \( (d(v, S_1), d(v, S_2), \ldots, d(v, S_k)) \). If all vertices of \( G \) have distinct color codes, then \( c \) is called a locating-chromatic \( k \)-coloring of \( G \) (k-locating coloring, in short). The locating-chromatic number \( \chi_L(G) \) of graph \( G \) is the smallest \( k \) such that \( G \) has a locating \( k \)-coloring.

Chartrand et al. [8] determined the locating-chromatic numbers for some well-known classes of graphs, namely paths, cycles, complete multipartite graphs and double stars. The locating-chromatic number of a path \( P_n \) is 3, for \( n \geq 3 \). The locating-chromatic number of a cycle \( C_n \) is 3 if \( n \) is odd and 4 otherwise. Furthermore, Chartrand et al. [9] studied the locating-chromatic number of trees in general. They showed that for any integer \( k \in \{3, 4, \ldots, n - 2, n\} \), there exists a tree of order \( n \) with locating-chromatic number \( k \). They also showed that no tree of order \( n \) exists with locating-chromatic number \( n - 1 \). Recently, Asmiati et al. [1, 3], determined the locating-chromatic number for an amalgamation of stars and firecracker graphs.

Some authors also consider the locating-chromatic number for graphs produced by a graph operation. For instances, Baskoro and Purwasih [4] determined the locating-chromatic number for the corona product of two graphs. Behtoei and Omoomi obtained the locating-chromatic number for the Cartesian product of graphs [6] and for the join product of graphs [7]. In particular, they also obtained the locating chromatic number of the fans, wheels and friendship graphs. In [5], Behtoei and Omoomi also considered the locating chromatic number of Kneser graphs.

Certainly, the only graph with locating-chromatic number \( n \) is a multipartite complete graph with \( n \) vertices. Furthermore, Chartrand et al. [9] also characterized all graphs on \( n \) vertices whose locating-chromatic number is \( n - 1 \). In the same paper, they showed that if \( G \) is a connected graph of order \( n \geq 5 \) containing an induced subgraph \( F \in \{2K_1 \cup K_2, P_2 \cup P_3, H_1, H_2, H_3, P_2 \cup K_3, P_2, C_5, C_5 + e\} \), then \( \chi_L(G) \leq n - 2 \). All graphs of order \( n \) with locating-chromatic number 3 are still not fully characterized. We know that \( P_n, n \geq 3 \), is an example of a graph with locating-chromatic number 3. Recently, we characterized all graphs containing a cycle with the locating-chromatic number 3 [2]. In this paper, we will determine all trees with locating-chromatic number 3. Therefore, this paper will complete the characterization of all graphs with locating-chromatic number 3. We also give a family of trees with locating-chromatic number 4.

2. Basic Properties

In this section, we give some definitions and basic properties related to graphs with locating-chromatic number 3. Let \( c \) be a locating \( k \)-coloring on graph \( G(V, E) \). Let \( \Pi = \{S_1, S_2, \ldots, S_k\} \) be the partition of \( V(G) \) induced by \( c \). A vertex \( v \in G \) is called a dominant vertex if \( d(v, S_i) = 1 \) if \( v \notin S_i \). A path connecting two dominant vertices in \( G \) is called a clear path if all of its internal
vertices are not dominant. Then, we have the following lemma as a direct consequence of the definition of dominant vertices.

**Lemma 2.1.** [2] Let $G$ be a graph with $\chi_L(G) = k$. Then, there are at most $k$ dominant vertices in $G$ and all of them must receive different colors.

**Lemma 2.2.** [3] Let $G$ be a graph with $\chi_L(G) = 3$. Then, the length of any clear path in $G$ is odd.

**Lemma 2.3.** [3] Let $G$ be a connected graph with $\chi_L(G) = 3$. If $G$ contains three dominant vertices, then these three dominant vertices must lie in a path.

3. Characterization

Consider two specific caterpillars $C(2, 2, 2)$ and $C(2, 1, 0, \ldots, 0, 1, 2)$, for any odd $t$, as depicted in the left side of Figure 1. Let $G_1$ be the subdivision of $C(2, 2, 2)$ on six pendant edges in $k_1, k_2, \ldots, k_6$ times respectively, where $k_i \geq 1$. Let $G_2$ be the subdivision of $C(2, 1, 0, \ldots, 0, 1, 2)$ on six pendant edges in $k_1, k_2, \ldots, k_6$ times respectively, where $k_i \geq 1$.

![Figure 1. The specific caterpillars and their subdivisions.](image)

Let $T$ be the class of all trees whose locating-chromatic number is 3. In this section, we characterize all trees which are members of $T$.

**Lemma 3.1.** Let $T \in T$. The color code of any vertex of $T$ is $(c_1, c_2, c_3)$ such that $\{c_1, c_2, c_3\} = \{0, 1, k\}$ where $k \geq 1$.

**Proof.** Let $x \in T$ and without loss of generality assume $c(x) = 1$. Since the neighbor of $x$ must have a different color then the color code of $x$ is either $(0, 1, k)$ or $(0, k, 1)$, where $k \geq 1$. □
For any integer \( k \geq 1 \), a tree \( T \in \mathcal{T} \) is called \( k \)-maximal if \( T \) has all possible color codes with \( k \) is the maximum ordinate. In this case, there is a locating coloring of \( T \) such that each color class in \( T \) has exactly \( 2k - 1 \) vertices. For example, graph \( C(2, 2, 2) \) is 2-maximal, since this graph has a locating coloring with each color class having 3 vertices. It can be verified that a path on \( 6k - 3 \) vertices is \( k \)-maximal. However, not all tree on \( 6k - 3 \) vertices are \( k \)-maximal.

**Lemma 3.2.** Let \( T \in \mathcal{T} \). Every vertex \( x \) of \( T \) has degree at most 4.

**Proof.** To the contrary, assume there is a vertex \( x \) with \( d(x) \geq 5 \). Let \( a_1, a_2, a_3, a_4, a_5 \) be the neighbors of \( x \). Let \( c \) be a locating 3-coloring of \( T \). Assume \( c(x) = 1 \) and so \( c(a_i) \) is either 2 or 3, for any \( i \in \{1, 2, 3, 4, 5\} \). If there are \( i \neq j \) such that \( c(a_i) \neq c(a_j) \) then there are at least three vertices \( a_i \) with the same color, say color 2. Thus, two of these vertices will have the same color code, a contradiction. Now, assume that the colors of all vertices \( a_i \) are the same, say \( c(a_i) = 2 \), for all \( i \in \{1, 2, \cdots, 5\} \). Let \( r = \min\{d(a_i), S_3\} \), where \( S_3 \) is the partition class consisting of all vertices whose color is 3. Then, the possible color codes for vertices \( a_i \) are \((1, 0, r), (1, 0, r + 1), \) or \((1, 0, r + 2) \). Therefore, we will have two vertices \( a_i \) with the same color code, a contradiction.

From now on, let \( T \in \mathcal{T} \). By Lemma 2.1, \( T \) has at most three dominant vertices. Clearly, if \( T \) is either a path \( P_3 \), or \( P_4 \), a double star \( S_{1,2} \) or \( S_{2,2} \), then \( T \) has a locating coloring such that \( T \) has only one or two dominant vertices. If \( T \) is not isomorphic to one of them, \( T \) must have exactly three dominant vertices. Let \( x, y, z \) be their dominant vertices. Up to isomorphism, assume that \( c(x) = 1 \), \( c(y) = 2 \) and \( c(z) = 3 \). By Lemma 2.3, there are two clear paths in \( T \): one connecting vertices \( x \) to \( y \), and the other one connecting \( y \) to \( z \). Let the two paths be \( xP_y := (x = u_0, u_1, u_2, \cdots, u_{r-1}, u_r = y) \) and \( yP_z := (y = v_0, v_1, v_2, \cdots, v_{s-1}, v_s = z) \) with \( r, s \) odd. Then, \( c(u_i) = 1 \) for even \( i \) and 2 for odd \( i \); and \( c(v_i) = 2 \) for even \( i \) and 3 for odd \( i \). Otherwise, there would be the fourth dominant vertex in \( T \). Since \( x \) is a dominant vertex in \( T \), then \( d(x) \geq 2 \). Therefore, there must be a neighbor of \( x \) (other than \( u_1 \)), say \( a \) with \( c(a) = 3 \). Similarly, there must be a vertex \( b \), a neighbor of \( z \) (other than \( v_{s-1} \)), with \( c(b) = 1 \). So, we have a path \( P \), where \( P = (a, x, u_1, u_2, \cdots, u_{r-1}, u_r = y, v_1, v_2, \cdots, v_{s-1}, v_s = z, b) \), with \( r, s \) odd. If \( r, s > 1 \) then define \( u^* = u_{\lceil \frac{r}{2} \rceil}, u^{**} = u_{\lceil \frac{r}{2} \rceil + 1}, v^* = v_{\lceil \frac{s}{2} \rceil}, \) and \( v^{**} = v_{\lfloor \frac{s}{2} \rfloor + 1} \).

**Lemma 3.3.** If \( r = s = 1 \) then \( 1 \leq d(a) \leq 2, 2 \leq d(x) \leq 3, 2 \leq d(y) \leq 4, 2 \leq d(z) \leq 3, \) and \( 1 \leq d(b) \leq 2 \). Furthermore, every vertex \( w \in V(T) \setminus P \) has degree at most 2 and is connected by a unique shortest path to one of \( \{a, x, y, z, b\} \).

**Proof.** For a contradiction, assume \( d(a) > 3 \) then two neighbors of \( a \) other than \( x \) will receive color 1. However, this implies that these neighbors will have the same color code, a contradiction. Therefore, \( d(a) \leq 2 \). Similarly, we also conclude that \( d(b) \leq 2 \). Next, since \( x \) is a dominant vertex, then \( d(x) \geq 2 \). Now, assume that \( d(x) \geq 4 \). Then, two of the neighbors of \( x \) will have the same color codes, a contradiction. Therefore, \( 2 \leq d(x) \leq 3 \). Similarly, we have that \( 2 \leq d(z) \leq 3 \). Since \( y \) is a dominant vertex and by Lemma 3.2 we have that \( 2 \leq d(y) \leq 4 \).

Let \( w \in V(T) \setminus P \). Since \( T \) is a tree, then there exists a unique shortest path \( L \) connecting \( w \) to a vertex of \( P \). If \( d(w) \geq 3 \) then there are two neighbors of \( w \), say \( w_1 \) and \( w_2 \), which are not in \( L \). Since \( \chi_L(T) = 3 \) and \( x, y \) and \( z \) are the dominant vertices of \( T \) then the color codes of \( w_1 \)
and \( w_2 \) will be the same, a contradiction. Therefore, every vertex \( V(T) \setminus P \) must have degree at most 2. The path \( L \) which connects \( w \) to \( P \) is unique (since \( T \) is a tree) and goes through one of \( \{a, x, y, z, b\} \).

\[ \square \]

**Lemma 3.4.** If \( r = 1 \) and \( s > 1 \) then \( 1 \leq d(a) \leq 2, 2 \leq d(x) \leq 3, 2 \leq d(y) \leq 3, 2 \leq d(v^*) \leq 3, 2 \leq d(v^{**}) \leq 3, d(z) = 2, \) and \( 1 \leq d(b) \leq 2. \) All the other internal vertices \( v_i \) in \( P \) have degree 2. Furthermore, every vertex \( w \in V(T) \setminus P \) has degree at most 2 and is connected by a unique shortest path to one of \( \{a, x, y, b, v^*, v^{**}\} \).

**Proof.** To show \( 1 \leq d(a) \leq 2, 2 \leq d(x) \leq 3, 2 \leq d(y) \leq 3, \) and \( d(b) \leq 2, \) we use a similar argument as in Lemma 3.3. Next, since \( v^* \) and \( v^{**} \) are internal vertices in \( P, \) then \( d(v^*), d(v^{**}) \geq 2. \) Assume \( d(v^*) \geq 4. \) Since \( v^* \) is not a dominant vertex then its two neighbors not in \( P \) will receive the same color. This implies that their color codes are the same, a contradiction. Therefore, \( d(v^*) \leq 3. \) Similarly, we have \( d(v^{**}) \leq 3. \) If \( z \) has the third neighbor \( z_1 \) then the color code of \( z_1 \) will be the same as the color code of either \( v_{s-1} \) or \( b. \) Therefore, \( d(z) = 2. \) Now, let \( v_i \) be an internal vertex in \( yP_z \) other than \( v^* \) or \( v^{**}. \) Assume \( d(v_i) \geq 3. \) Since all the neighbors of \( v_i \) are not dominant vertices, they will receive the same color. Thus, two of them will have the same color code, a contradiction. Therefore, \( d(v_i) = 2 \) for any \( v_i \) other than \( v^* \) and \( v^{**}. \)

Let \( w \in V(T) \setminus P. \) Since \( T \) is a tree, then there exists a unique shortest path \( L \) connecting \( w \) to a vertex of \( P. \) If \( d(w) \geq 3 \) then there are two neighbors of \( w, \) say \( w_1, w_2, \) which are not in \( L. \) Since \( \chi_L(T) = 3 \) and \( x, y, \) and \( z \) are the dominant vertices of \( T \) then the color codes of \( w_1 \) and \( w_2 \) will be the same, a contradiction. Therefore, every vertex \( V(T) \setminus P \) has degree at most 2. The path \( L \) which connects \( w \) to \( P \) is unique (since \( T \) is a tree) and goes through one of \( \{a, x, y, b, v^*, v^{**}\}. \)

\[ \square \]

**Lemma 3.5.** If \( r > 1 \) and \( s > 1 \) then \( 1 \leq d(a) \leq 2, d(x) = d(y) = d(z) = 2, 2 \leq d(u^*) \leq 3, 2 \leq d(u^{**}) \leq 3, 2 \leq d(v^*) \leq 3, 2 \leq d(v^{**}) \leq 3, \) and \( d(b) \leq 2. \) All the other internal vertices in \( P \) have degree 2. Furthermore, every vertex \( w \in V(T) \setminus P \) has degree at most 2 and is connected by a unique shortest path to one of \( \{a, b, u^*, u^{**}, v^*, v^{**}\}. \)

**Proof.** The proof is similar as in the proof of Lemma 3.4.

\[ \square \]

**Theorem 3.1.** If \( T \in \mathcal{T} \) and \( T \) has maximum number of vertices of degree higher than 2, then \( T \) is isomorphic to either \( G_1 \) or \( G_2. \)

**Proof.** Let \( T \in \mathcal{T}. \) By Lemma 2.1, \( T \) contains at most three dominant vertices. If \( T \) is either a path \( P_3, \) or \( P_4, \) a double star \( S_{1,2} \) or \( S_{2,2}, \) then \( T \) has a locating coloring such that \( T \) has only one or two dominant vertices. If \( T \) is not isomorphic to one of these four graphs above, then \( T \) will have a locating coloring with exactly three dominant vertices. Let \( x, y, z \) be such vertices. Then by Lemma 2.3, there are two clear paths, namely: \( xP_y = (x = u_0, u_1, u_2, \cdots, u_{r-1}, u_r = y), \) \( yP_z = (y = v_0, v_1, v_2, \cdots, v_{s-1}, v_s = z), \) with \( r, s \) odd.

If \( r = s = 1 \) then by Lemma 3.3, \( T \) will have maximum number of vertices of degree higher than 2 if there are two paths attached to \( y \) and one path attached to each vertex of \( a, x, z, \) and \( b, \) as depicted in Figure 2(i). Now, define a coloring \( c : V(T) \rightarrow \{1, 2, 3\} \) such that:

1. \( c(a) = 3, c(x) = 1, c(y) = 2, c(z) = 3, c(b) = 1; \)
2. The colors of vertices of the path \( L_0 \) attached to \( a \) are 1 and 3 alternately;
3. The colors of vertices of the path \( L_x \) attached to \( x \) are 2 and 1 alternately;
4. The colors of vertices of the first path \( L_y^1 \) attached to \( y \) are 1 and 2 alternately;
5. The colors of vertices of the second path \( L_y^2 \) attached to \( y \) are 3 and 2 alternately;
6. The colors of vertices of the path \( L_z \) attached to \( z \) are 2 and 3 alternately;
7. The colors of vertices of the path \( L_b \) attached to \( b \) are 3 and 1 alternately.

The color codes of all vertices of \( L_0 \) are \((1, \text{even}, 0)\) or \((0, \text{odd}, 1)\). The color codes of all vertices of \( L_x \) are \((0, 1, \text{odd})\) or \((1, 0, \text{even})\). The color codes of all vertices of \( L_y \) are \((0, 1, \text{even})\) or \((1, 0, \text{odd})\). The color codes of all vertices of \( L_z \) are \((\text{even}, 0, 1)\) or \((\text{odd}, 1, 0)\). The color codes of all vertices of \( L_b \) are \((0, \text{even}, 1)\) or \((1, \text{odd}, 0)\). Therefore, all the color codes are different. Thus, \( c \) is a locating-coloring on \( T \). Since 3 is the smallest possible number of colors then \( \chi_L(T) = 3 \). In this case, \( T \) is isomorphic to \( G_1 \).

If \( r = 1 \) and \( s > 1 \) then by Lemma 3.4, \( T \) will have maximum number of vertices of degree higher than 2 if there is one path attached to vertices \( a, x, y, v^*, v^{**} \) and \( b \) each as depicted in Figure 2(ii). Now, define a coloring \( c : V(T) \to \{1, 2, 3\} \) such that:

1. \( c(a) = 3, c(x) = 1, c(y) = 2, c(z) = 3, c(b) = 1; \)
2. The colors of vertices of the path \( L_0 \) attached to \( a \) are 1 and 3 alternately;
3. The colors of vertices of the path \( L_x \) attached to \( x \) are 2 and 1 alternately;
4. The colors of vertices of the path \( L_y \) attached to \( y \) are 1 and 2 alternately;
5. The colors of internal vertices \( v, s \) are 3 and 2 alternately;
6. The colors of vertices of the path \( L_{v^*} \) attached to \( v^* \) are 2 and 3 alternately;
7. The colors of vertices of the path \( L_{v^{**}} \) attached to \( v^{**} \) are 3 and 2 alternately;
8. The colors of vertices of the path \( L_b \) attached to \( b \) are 3 and 1 alternately.

Then, it can be verified that the color codes of all vertices in \( T \) are distinct. Therefore, \( c \) is a locating-coloring on \( T \). Since 3 is the smallest possible number of colors then \( \chi_L(T) = 3 \). In this case, \( T \) is isomorphic to \( G_2 \).

If \( r > 1 \) and \( s > 1 \) then by Lemma 3.5, \( T \) will have maximum number of vertices of degree higher than 2 if there is one path attached to \( a, u^*, u^{**}, v^*, v^{**} \) and \( b \) each, as depicted in Figure 1(iii). By defining a similar coloring \( c \) we can obtain \( \chi_L(T) = 3 \). In this case, \( T \) is isomorphic to \( G_2 \). \( \square \)

**Theorem 3.2.** A tree \( T \) has the locating-chromatic number 3 if and only if \( T \) is either a path \( P_3 \) or \( P_4 \), a double star \( S_{1,2} \) or \( S_{2,2} \) or a subtree containing a path \( P \) of either \( G_1 \) or \( G_2 \).

**Proof.** If \( T \) is either \( P_3 \), \( P_4 \), \( S_{1,2} \) or \( S_{2,2} \) then clearly it has locating-chromatic number 3. Now, let \( T^* \) be a subtree of either \( G_1 \) or \( G_2 \), and it contains a path \( P \) of length at least 4, as illustrated in Figure 2. Then, by using the coloring \( c \) in Theorem 3.1 restricted to the subtree \( T^* \), we obtain that all the color codes are different. Therefore, \( \chi_L(T^*) = 3 \).

Conversely, let \( T \) be a tree with locating-chromatic number 3. If the diameter of \( T \) is \( \leq 3 \) then \( T \) must be either \( P_3 \), \( P_4 \), \( S_{1,2} \) or \( S_{2,2} \). Now, if the diameter of \( T \) is \( \geq 4 \) then by Lemma 2.1 \( T \) has at
Figure 2. A tree $T \in \mathcal{T}$ with maximum number of vertices of degree higher than 2 and it contains a path $P = \{a, x, u_1, \cdots, u_r = y, v_1, \cdots, v_s = z, b\}$.
most 3 dominant vertices. Clearly, if \( T \) is not isomorphic to one of these four graphs above, then \( T \) will have a locating coloring such that \( T \) has exactly three dominant vertices. By Lemma 2.3, the three dominant vertices must lie in a path. This path must be of length at least 4. By Theorem 3.1, we conclude that \( T \) must be a subtree of one of the trees in Figure 2. As a consequence, \( T \) is a subtree of either \( G_1 \) or \( G_2 \).

In the following theorems, we will give an infinite number of trees with locating-chromatic number 4 constructed from the trees with locating-chromatic number 3.

**Theorem 3.3.** Let \( T' \) be a tree constructed from either \( G_1 \) or \( G_2 \) by attaching a path of arbitrary length to each vertex. Then, \( \chi_L(T') = 4 \).

**Proof.** Define a coloring \( c' : V(T') \to \{1, 2, 3, 4\} \) such that:

\[
c'(u) = c(u), \text{ for any } u \in T,
\]

where \( c \) is a coloring on \( T \) used in Theorem 3.1, and define the values of \( c' \) on any path \( L := (w = w_0, w_1, \ldots, w_t) \) attached to a vertex \( w \) as follows:

\[
c'(w_i) = c(w) \text{ for even } i, \text{ and } c'(w_i) = 4 \text{ for odd } i.
\]

We will show that \( c' \) is a locating-coloring. Let \( u, v \) be any two vertices of \( T' \) with \( c'(u) = c'(v) \). If \( u \) and \( v \) are in \( T \) then the color codes are distinct, since their color codes are derived from the previous color codes (under \( c \)) by adding the fourth ordinate with entry 1. If \( u \in T \) and \( v \not\in T \) then \( d(v, S) > d(u, S) \), where \( S \) is either \( S_1, S_2 \) or \( S_3 \), with \( S_i \) being the set of vertices receiving color \( i \) under \( c' \). Now, let \( u \not\in T \) and \( v \not\in T \). If \( u \) and \( v \) are in the same path attached to vertex \( w \) then \( d(u, S) \neq d(v, S) \) with \( S \) being either \( S_1, S_2 \) or \( S_3 \). Now, let \( u \) be in a path \( L_1 \) attached to \( w' \) and \( v \) be in a path \( L_2 \) attached to \( w'' \). If \( c'(w') = c'(w'') \) then the color codes of \( u \) and \( v \) are different since the color codes of \( w' \) and \( w'' \) are different. If \( c'(w') \neq c'(w'') \) then \( d(u, S) = 1 < d(v, S) \) with \( S \) being the partition class containing vertex \( w' \). Therefore, all vertices in \( T' \) have distinct color codes. Thus, \( c' \) is a locating-coloring on \( T' \). Since 4 is the smallest possible number of colors (by Theorem 3.1) then \( \chi_L(T') = 4 \). \( \square \)

**Theorem 3.4.** Let \( T' \) be a tree constructed in Theorem 3.3. Every subtree of \( T' \) which is not a subtree of \( G_1 \) or \( G_2 \) has locating-chromatic number 4.

**Proof.** A direct consequence of Theorem 3.3. \( \square \)

To conclude this paper, we present an open problem related to the locating-chromatic number of graphs.

**Problem 1.** Characterize all graphs of order \( n \geq 4 \) with locating-chromatic number 4.

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