Graphs with coloring redundant edges

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Abstract

A graph edge is \(d\)-coloring redundant if the removal of the edge does not change the set of \(d\)-colorings of the graph. Graphs that are too sparse or too dense do not have coloring redundant edges. Tight upper and lower bounds on the number of edges in a graph in order for the graph to have a coloring redundant edge are proven. Two constructions link the class of graphs with a coloring redundant edge to the \(K_4\)-free graphs and to the uniquely colorable graphs. The structure of graphs with a coloring redundant edge is explored.

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1. Preliminaries

As usual in graph coloring (see for instance [3]), we focus on simple connected graphs; \(\chi(G)\) denotes the chromatic number of a graph \(G\), i.e. the smallest number of colors needed to color \(G\). For convenience, we number the colors from 1 upwards. We use \(col(v)\) to denote the color of node \(v\) in a particular coloring. \(G(V,E)\) denotes a graph with node set \(V\) and edge set \(E\). We denote by \(G_{ab}\) the graph \(G(V,E \setminus \{(a,b)\})\), and by \(G^{ab}\) the graph \(G(V \cup \{a,b\}, E \cup \{(a,b)\})\). In the sequel, larger or smaller graph has to be understood in terms of the number of the edges.

We make use of \textit{complete \(d\)-partite graphs} denoted by \(K_{a_1,a_2,...,a_d}\) with (for convenience) \(a_i \geq a_{i+1}\). The \textit{Turán graphs} \(T(n,d)\), introduced in [6], can be characterized as any \(K_{a_1,a_2,...,a_d}\) for which

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\((a_1 - a_d) \leq 1\) and \(n = \sum_{i=1}^{d} a_i\). An alternative characterization is that \(T(n, d)\) is the largest \(d\)-partite graph with \(n\) nodes.

**Definition 1.** An edge \((a, b)\) in a connected graph \(G\) is \(d\)-**coloring redundant** \((d\text{-CR})\) if \(G\) is \(d\)-colorable and every \(d\)-coloring of \(G_{ab}\) assigns different colors to \(a\) and \(b\).

The set of graphs with \(n\) nodes and a \(d\)-coloring redundant edge is denoted by \(GCRE(n, d)\): we are in particular interested in the size (the number of edges) of the graphs for combinations of \(n\) and \(d\). The removal of a \(d\)-CR edge \((a, b)\) of \(G\) does not change the set of \(d\)-colorings, or otherwise said: any \(d\)-coloring of \(G_{ab}\) is a \(d\)-coloring of \(G\).

**Lemma 1.1.** If \(G \in GCRE(n, d)\) then \(d = \chi(G)\).

*Proof.* From the definition it follows that \(d \geq \chi(G)\). Suppose \(d > \chi(G)\). Let \((a, b)\) be a \(d\)-CR edge. Let \(C\) denote a color number larger than \(\chi(G)\). Consider a \(d\)-coloring of \(G\) constructed as follows: first construct a \(\chi(G)\)-coloring of \(G_{ab}\), and then change \(\text{col}(a)\) and \(\text{col}(b)\) into \(C\). This results in a \(d\)-coloring of \(G_{ab}\) in which \(a\) and \(b\) have the same color, which contradicts the choice of \((a, b)\). Hence the lemma follows. \(\square\)

This lemma allows us to drop the reference to the chromatic number \(d\), and simply say \(G \in GCRE\), or \((a, b)\) is a CR edge. We state two lemma’s without a proof. The first one says that the removal of a CR edge does not change the chromatic number.

**Lemma 1.2.** If \(G\) has a CR edge \((a, b)\), then \(\chi(G) = \chi(G_{ab})\). \(\square\)

Lemma 1.2 has an analogue in which an edge is added, and which is useful while constructing larger \(GCRE\).

**Lemma 1.3.** If \(G \in GCRE(n, d)\), then either \(G_{ab} \in GCRE(n, d)\) or \(\chi(G) < \chi(G_{ab})\). \(\square\)

From Lemma 1.3, it follows that if \(G'\) is a subgraph of a connected graph \(G\) with \(\chi(G') = \chi(G)\) and \(G' \in GCRE\), then also \(G \in GCRE\).

**Lemma 1.4.** For every \(k \geq 1\), \(G = K_{k,1,1,...,1}\) is not in \(GCRE\).

*Proof.* Let the natural partition of \(G\)’s nodes be \(\{x_1, \ldots, x_k\}, \{a_2\}, \{a_3\}, \ldots, \{a_d\}\). An edge \((a_i, a_j)\) (with \(i \neq j\)) is not CR, because \(G_{a_ia_j}\) can be colored with less than \(d\) colors (see Lemma 1.2). No edge \((x_i, a_j)\) is CR, because \(G_{x_ia_j}\) can be colored while giving the same color to \(x_i\) and \(a_j\). \(\square\)

**Theorem 1.1.** Let \(\{A_1, \ldots, A_d\}\) be a partition of \(n\) nodes so that \(|A_i| = a_i, a_i \geq a_{i+1}\) and \(\sum a_i = n\). Then \(G = K_{a_1,a_2,...,a_d} \in GCRE(n, d)\) if and only if \(a_2 \geq 2\). The only CR edges are the edges between \(A_i\) and \(A_j\) for which \(a_i > 1\) and \(a_j > 1\).

*Proof.* An edge between the nodes in \(A_i\) and \(A_j\) for which \(a_i = a_j = 1\) cannot be CR because without this edge, the node in \(A_i\) can have the same color as the node in \(A_j\). Similarly, if \(a_i = 1\) and \(a_j > 1\), the removal of an edge between \(A_i\) and \(A_j\) allows a \(d\)-coloring with the same color for the two involved nodes, so such edges cannot be CR.
This leaves edges between $A_i$ and $A_j$ ($i < j$) each with at least two nodes. Name these selected nodes $v_{i,1}, v_{i,2}$ (both in $A_i$) and $v_{j,1}, v_{j,2}$ (in $A_j$). Let $v_k$ be nodes selected from $A_k$ for $k \notin \{i, j\}$. $G$ contains as a subgraph the $d$-clique with nodes \{v_1, v_2, ..., v_{i-1}, v_{i,1}, v_{i,2}, ..., v_{j-1}, v_{j,1}, v_{j,2}, v_{j+1}, ..., v_d\} and the $d$-clique with nodes \{v_1, v_2, ..., v_{i-1}, v_{i,1}, v_{i,2}, ..., v_{j-1}, v_{j,1}, v_{j,2}, v_{j+1}, ..., v_d\}.

As a consequence, the nodes $v_{i,1}$ and $v_{i,2}$ have the same color in any $d$-coloring, which implies that the edges $(v_{i,1}, v_{j,2})$ and $(v_{i,2}, v_{j,2})$ are coloring redundant. Figure 1 exemplifies the situation for $d = 4$, $i = 2$, $j = 3$: a dashed line between two node sets means that all nodes of one set are connected by an edge to all nodes of the other set.

By symmetry, all edges between sets with at least two nodes are CR, and no other edges are.

Since $T(n,d)$ is a complete $d$-partite graph, we can conclude that $T(n,d) \in GCRE(n,d)$ if $(n \geq d + 2)$ and $d \geq 2$.

2. The results

The next subsections explore the size and the structure of elements of $GCRE(n,d)$ for all values of $n$ and $d$.

2.1. Maximal $GCRE(n,d)$

Theorem 2.1. $GCRE(n,d) = \emptyset$ for $n < d + 2$, and the maximal elements of $GCRE(n,d)$ with $n \geq d + 2$ are the Turán graphs $T(n,d)$.

Proof. A largest element - i.e. one with the highest number of edges - $G \in GCRE(n,d)$ has the following properties:

- it is $d$-partite (because $\chi(G) = d$)
- adding any new edge results in a graph with chromatic number $(d + 1)$ (see Lemma 1.3) because $G$ is maximal

It follows that such a largest graph $G$ equals a $K_{a_1, ..., a_d}$. A $K_{a_1, ..., a_d}$ which gives the maximal number of edges under the restriction that $\sum_{i=1}^{d} a_i = n$ is the Turán graph $T(n,d)$. Lemma 1.4 implies that $T(n,d)$ is indeed in $GCRE(n,d)$ for $n \geq d + 2$, and that otherwise $GCRE(n,d) = \emptyset$.

As a conclusion, we can state that up to isomorphism, there is only one largest element in $GCRE(n,d)$, and its number of edges is $\left\lfloor \frac{(d-1)n^2}{2d} \right\rfloor$. 

Figure 1: $K_{2,2,2}$. 

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2.2. Minimal GCRE(n,d)
For a graph $G$ with edge $(a, b)$, we denote by $G_{a=b}$ the graph in which $(a, b)$ is contracted.

**Lemma 2.1.** Let $G \in GCRE(n, d)$ with $e$ edges, and let $(a, b)$ be one of its CR edges. Let $e$ be the number of edges in $G$. Then

- the number of nodes in $G_{a=b}$ equals $(n - 1)$
- the number of edges in $G_{a=b}$ is at most $(e - 1)$
- $G_{a=b}$ is connected
- $\chi(G_{a=b}) = d + 1$

**Proof.** The first three are trivial to prove. The last one can be proved as follows: suppose $G_{a=b}$ can be colored with $d$ colors, than this coloring can be lifted to a coloring of $G_{ab}$ in which $a$ and $b$ have the same colors, which contradicts the fact that $(a, b)$ is CR. $G_{a=b}$ can be $(d + 1)$-colored as follows: color $G$ with $d$ colors, then assign to both $a$ and $b$ the $(d + 1)$th color and contract $(a, b)$.

**Theorem 2.2.** Let $e$ be the number of edges of $G \in GCRE(n, d)$. It follows that $n + \frac{d^2-d-2}{2} \leq e$. Moreover, there exist $G \in GCRE(n, d)$ for which equality holds.

**Proof.** For any connected graph with $n'$ nodes, $e'$ edges and chromatic number $d'$, the following inequality holds: $\frac{d'(d'-1)}{2} + (n' - d') \leq e'$.

Let $(a, b)$ be a CR edge of $G$. Then by using Lemma 2.1 for $G_{a=b}$, we can substitute $e'$, $d'$, $n'$ by $(e - 1), (d + 1), (n - 1)$ and derive:

$$\frac{d(d+1)}{2} + (n - 1) - (d + 1) \leq (e - 1) \text{ or equivalently } n + \frac{d^2-d-2}{2} \leq e$$

To prove the second part of the theorem, we establish one particular example: name the $n$ nodes $v_1, v_2, \ldots, v_n$. Connect the nodes as in Figure 2, i.e. the edges and their counts are

- $(v_1, v_2), (v_1, v_{d+2}), (v_{d+1}, v_2), (v_{d+1}, v_{d+2})$: 4 edges
- the ellipse represents a clique between the nodes $\{v_3, v_4, \ldots v_d\}$: $(d - 2)(d - 3)/2$ edges
- the multi-edges represent $(v_k, x)$ for $k = 3, 4, \ldots, d$ and $x \in \{v_1, v_2, v_{d+1}\}$: $3(d - 2)$ edges

![Figure 2: A GCRE(n, d) with $n + \frac{d^2-d-2}{2} = e$.](image-url)
• the nodes \(v_{d+2} \ldots v_n\) are connected amongst each other as \((v_i, v_{i+1})\) for \(i = (d + 2), (d + 3), \ldots, (n - 1)\): \((n - d - 2)\) edges

The graph is connected, has \(n\) nodes and its number of edges is \(4 + (d - 2)(d - 3)/2 + 3(d - 2) + (n - d - 2) = n + \frac{d^2 - d - 2}{2}\). Clearly, its chromatic number is \(d\). Finally, the edge \((v_1, v_{d+2})\) is CR because the \(d\)-cliques \(\{v_1, v_2, v_3, \ldots, v_d\}\) and \(\{v_{d+1}, v_2, v_3, \ldots, v_d\}\) force \(v_1\) and \(v_{d+1}\) to have the same color in any \(d\)-coloring. The reasoning is similar to the one in Theorem 1.1.

2.3. **Intermediate GCRE\((n,d)\)**

**Theorem 2.3.** For all \(n,d,e:\ d \geq 2, n \geq d + 2, n + \frac{d^2 - d - 2}{2} \leq e \leq \lfloor \frac{(d-1)n^2}{2d} \rfloor\), there exists a \(G \in \text{GCRE}(n,d)\) such that \(G\) has exactly \(e\) edges.

**Proof.** Consider the graph constructed in Theorem 2.2. Define the sets \(V_i, i = 1..d\) such that \(V_i = \{v_{i+k \cdot d} | k = 0..\lfloor \frac{n-1}{d} \rfloor\}\): these sets form an equitable partition of the nodes. The graph does not contain any edge between nodes of the same \(V_i\). Add one by one as many edges as possible between nodes in different \(V_i\). This keeps the chromatic number equal to \(d\), and from Lemma 1.3, all intermediate graphs are in \(\text{GCRE}(n,d)\). Thanks to the choice of the partition, when the maximal amount of edges is added, the result is \(T(n,d)\).

2.4. **The structure of graphs in GCRE**

Let \(G_{\backslash\{a,b\}}\) denote the graph \(G\) from which \(a\) and \(b\) and all their edges are removed. We use \(\delta_X\) for the degree of a node in a graph \(X\).

**Lemma 2.2.** If \(G\) is \(\text{GCRE}(n,d)\) and \((a,b)\) is CR in \(G\), then \(\chi(G_{\backslash\{a,b\}}) = d\).

**Proof.** Suppose that there exists a \((d - 1)\)-coloring of \(G_{\backslash\{a,b\}}\), then assign the \(d\)th color to \(a\) and \(b\), and get a \(d\)-coloring of \(G_{ab}\) in which \(a\) and \(b\) have the same color, but this contradicts the choice of \((a,b)\).

**Lemma 2.3.** Let \(G\) be connected, with \(\chi(G) = d\), \(n\) nodes and the edge \((a,b)\). \((a,b)\) is CR in \(G\) if and only if every \(d\)-coloring col of \(G_{ab}\) satisfies

\[|\text{col}(N_a) \cup \text{col}(N_b)| = d \quad \text{(or equivalently } col(N_a) \cap \overline{\text{col}(N_b)} = \emptyset)\]

where \(N_a\) (resp. \(N_b\)) is the set of neighbors of \(a\) (resp. \(b\)) in \(G_{ab}\), and \(\overline{A}\) means the complement with respect to the \(d\) available colors.

**Proof.** \((\Rightarrow)\) Suppose that for some \(d\)-coloring of \(G_{ab}\), \(\overline{\text{col}(N_a)} \cap \text{col}(N_b)\) contains at least one color, say \(C\). One can then change the color of \(a\) and \(b\) to \(C\), contradicting the choice of \((a,b)\).

\((\Leftarrow)\) Suppose a \(d\)-coloring of \(G_{ab}\) exists that gives the same color \(C\) to \(a\) and \(b\). That implies that \(C \in \text{col}(N_a) \cap \overline{\text{col}(N_b)}\), which violates the assumption.

Lemma 2.3 implies that for a CR edge \((a,b)\) in \(G\), \(|N_a \cup N_b| \geq d\), and consequently \(\delta_{G_{ab}}(a) + \delta_{G_{ab}}(b) \geq |N_a \cup N_b| \geq d\).
The general structure of $G \in GCRE(n, d)$ with CR edge $(a, b)$ becomes more clear now. As in Figure 3, $G$ consists of a subgraph with the same chromatic number as $G$. The vertices $a$ and $b$ are connected to that subgraph with at least $d$ neighbors. In the figure, the graph within the rectangle is $G_{\{a,b\}}$. $G_{\{a,b\}}$ does not need to be connected, but in some sense, one component is enough.

**Lemma 2.4.** If $G$ is minimal in $GCRE(n, d)$ and $(a, b)$ is CR in $G$, then $G_{\{a,b\}}$ is connected.

**Proof.** Let the components of $G_{\{a,b\}}$ be $S_1, \ldots, S_k$. Denote by $G_i$ the subgraph of $G$ induced by the vertices $a, b$ and the nodes of $S_i$. Suppose $(a, b)$ is not CR in any $G_i$, then each $G_{iab}$ has a $d$-coloring $C_i$ in which $col_i(a) = col_i(b)$. Rename the colors so that $col_i(a) = col_j(a)$ for all $i, j$: this results in a $d$-coloring of $G_{ab}$ in which $a$ and $b$ have the same color, which is impossible. \hfill \qed

3. $K_4$-free, and uniquely colorable $GCRE(n, d)$

$K_4$-free $GCRE(n, d)$. We show a general construction of a $GCRE(n, d)$ without a 4-clique for every $d > 3$. The basis for this construction is Mycielski’s Theorem [5] and the so called *Iterated Mycielskians* $M_i$ which are a sequence of triangle-free graphs with chromatic number $i$. We start from such a graph $M_i$ with $i = \chi(M_i) \geq 3$ and let $m$ be the number of its nodes. Construct the graph $G$ that has $M_i$ as a subgraph and the following additional nodes and edges (as shown in Figure 4):

- three new nodes named $a, b, x$
- $(a, z)$ and $(b, z)$ for all $z \in M_i$
- $(a, x)$ and $(b, x)$

We now prove that $G$ has no 4-clique and $G \in GCRE(m + 3, d)$ with $d = i + 1$:

- $\chi(G) = d$: $G$ clearly has a $d$-coloring, as a $(d-1)$-coloring of $M_i$ can be extended to $G$ by giving $a$ and $b$ both the $d^{th}$ color, and giving $x$ any color different from that. Now suppose that $G$ had a $(d-1)$-coloring: the restriction to $M_i$ could use only $(d-2)$ colors, since $col(a)$ must differ from all colors in $M_i$. So, $G$ has no $(d-1)$-coloring and $\chi(G) = d$

- edges $(a, x)$ and $(b, x)$ are CR: indeed, $a$ and $b$ have the same color in every $d$-coloring, so $(col(a) \neq col(x)) \Leftrightarrow (col(b) \neq col(x))$, which proves each of the two edges is CR

- $G$ has no 4-clique: suppose $G$ has a 4-clique $C_4$; since $M_i$ is triangle-free, $C_4$ contains at least one of $a,b$ or $x$; $x$ cannot be in $C_4$ because it has degree 2; $a$ and $b$ cannot be both in $C_4$ because there is no edge between them; so assume that $a \in C_4$; the restriction of $C_4$ to $M_i$ would then be a 3-clique; this contradicts the fact that $M_i$ is triangle-free
The construction applied to $M_3$ results in the graph in Figure 5: it has 17 edges and no 4-clique. It is not a minimal $GCRE(8, 4)$, but it is one of the three minimal $GCRE(8, 4)$ without a 4-clique.

A natural question in this context is: do triangle-free $GCRE(n, d)$ exist? Via a computer experiment, we found the graph in Figure 6: it is the only triangle-free graph in $GCRE(n, d)$ for all $n \leq 9$ and $d \geq 3$. The two dashed edges are CR.

**Uniquely colorable $GCRE$.** Any complete $d$-partite graph is uniquely colorable and Theorem 1.1 shows that infinitely many are also a $GCRE$. We give a general construction that turns every uniquely colorable graph into a uniquely colorable $GCRE$, without ending up necessarily with a complete $d$-partite graph.

In a uniquely colorable graph $G(V, E)$ with chromatic number $d$, one can partition $V$ in subsets $V_1, V_2, ..., V_d$ such that in every $d$-coloring, $\{col(v) \mid v \in V_i\}$ is a singleton for each $i$. From $G(V, E)$, we construct a new uniquely colorable graph $U$ whose nodes are $V \cup \{a, b\}$ ($a$ and $b$ are two new nodes) and whose edges consist of $E \cup \{(a, b)\} \cup \{(x, a) \mid x \in \bigcup_{i=1}^{d-1} V_i\} \cup \{(x, b) \mid x \in \bigcup_{i=2}^{d} V_i\}$.

One can check that $\chi(U) = d$, $(a, b)$ is CR, and that $U$ is uniquely colorable. The latter is a consequence of Theorem 4 from [4]: the partition according to the colors in any coloring is $V_1 \cup \{b\}, V_2, ..., V_d \cup \{a\}$. The additional edge $(a, b)$ retains the uniqueness of the coloring.

Figure 7 shows the construction starting from the uniquely colorable graph with the full lines: it is not in $GCRE$. The added edges are the dashed lines. The result is one edge short of $K_{3,3}$, and it is $GCRE$.

**4. Discussion and Future Work**

The motivation for this work comes from the study of redundant disequalities in the context of constraint programming: when transposed to the graph coloring context, a set of disequalities corresponds to the constraint graph with edges between disequal variables (the nodes), and a redundant
disequality (one implied by the others) corresponds to a CR edge. In [1] and [2], the redundant disequalities were fully classified for the Latin Square problem and for Sudoku. It seemed worthwhile to explore the graph context and this resulted in the current work. To sum up our results:

- the maximal number of edges in a $GCRE(n, d)$ is attained by the Turán graph $T(n, d)$; the number of edges equals $\left\lfloor \frac{(d-1)n^2}{2d} \right\rfloor$
- the minimal number of edges in any $GCRE$ with $n$ nodes and chromatic number $d$ equals $n + \frac{d^2-d-2}{2}$
- for each $e$, such that $n + \frac{d^2-d-2}{2} \leq e \leq \left\lfloor \frac{(d-1)n^2}{2d} \right\rfloor$, there exists a graph $G(V, E) \in GCRE(n, d)$ such that $e = |E|

This work has focussed solely on the existence of at least one CR edge. Ultimately, we want to understand graphs with many CR edges, and quantify that understanding. We would also like to develop (polynomial) algorithms that (approximately) complete the graph, i.e. to add as many CR edges as possible: this should benefit solving constraint satisfaction problems by typical constraint solvers. The observation in Lemma 2.3 could be of great value there. Finally, the extension of our work to list coloring is interesting, because it corresponds to constraint satisfaction problems in which the variables have different domains.

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