On the general sum-connectivity index of connected graphs with given order and girth

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Abstract

In this paper, we show that in the class of connected graphs $G$ of order $n \geq 3$ having girth at least equal to $k$, $3 \leq k \leq n$, the unique graph $G$ having minimum general sum-connectivity index $\chi_\alpha(G)$ consists of $C_k$ and $n - k$ pendant vertices adjacent to a unique vertex of $C_k$, if $-1 \leq \alpha < 0$. This property does not hold for zeroth-order general Randić index $0R_\alpha(G)$.

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Girth, pendant vertex, general sum-connectivity index, zeroth-order general Randić index, subadditive function, convex function, Jensen’s inequality
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1. Introduction

Let $G$ be a simple graph having vertex set $V(G)$ and edge set $E(G)$. Let $\mathcal{G}_n$ denote the set of connected graphs of fixed order $n$ and size $m \geq n$. The girth of a graph $G \in \mathcal{G}_n$ will be denoted $g(G)$. The degree of a vertex $u \in V(G)$ is denoted $d(u)$ and $N(u)$ is the set of vertices adjacent with $u$. If $d(u) = 1$ then $u$ is called pendant; a pendant edge is an edge containing a pendant vertex. The minimum and maximum degrees of $G$ are denoted $\delta(G)$ and $\Delta(G)$, respectively. For $A \subset E(G)$, $G - A$ denotes the graph deduced from $G$ by deleting the edges of $A$ and the graph obtained by the deletion of an edge $uv \in E(G)$ is denoted $G - uv$. Conversely, if $A \subset E(G)$, $G + A$ is the graph obtained from $G$ by adding the edges of $A$. If $x \in V(G)$, $G - x$ denotes the

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subgraph of $G$ obtained by deleting $x$ and its incident edges.  
For $n \geq 3$ and $3 \leq k \leq n$, let $C_{k,n-k}$ denote the graph of order $n$ consisting of a cycle $C_k$ and $n-k$ pendant edges attached to a unique vertex of $C_k$. For other notations in graph theory, we refer [1].  
The general sum-connectivity index of graphs was proposed by Zhou and Trinajstić [10]. It is denoted by $\chi_\alpha(G)$ and defined as  
$$
\chi_\alpha(G) = \sum_{uv \in E(G)} (d(u) + d(v))^{\alpha},
$$
where $\alpha$ is a real number. A particular case of the general sum-connectivity index is the harmonic index, denoted by $H(G)$ and defined as  
$$
H(G) = \sum_{uv \in E(G)} \frac{2}{d(u) + d(v)} = 2\chi_{-1}(G).
$$

The zeroth-order general Randić index, denoted by $^0R_\alpha(G)$ is defined as  
$$
^0R_\alpha(G) = \sum_{u \in V(G)} d(u)^\alpha,
$$
where $\alpha$ is a real number. For $\alpha = 2$ this index is also known as first Zagreb index (see [4]).  
For $-1 \leq \alpha < 0$ Du, Zhou and Trinajstić [2] showed that among the set of $n$-vertex unicyclic graphs with $n \geq 5$, $C_{3,n-3}$ is the unique graph with the minimum general sum-connectivity index and Tomescu and Kanwal [6] showed that in the same set of graphs having girth $k \geq 4$ the unique extremal graph is $C_{k,n-k}$. Zhong [9] proved that in the set of connected graphs of order $n$ and $m$ edges, where $m \geq n$, with girth $g(G) \geq k$ ($3 \leq k \leq n$), minimum harmonic index $H(G)$ is reached only for $C_{k,n-k}$. Other extremal properties of the general sum-connectivity index for trees were proposed in [3, 5].  
In this paper, we study the minimum general sum-connectivity index $\chi_\alpha(G)$ in the class of connected graphs $G$ of fixed order $n \geq 3$ and size $m \geq n$ with girth $g(G) \geq k$. Theorem 3.1 extends the above result of Zhong for every $-1 \leq \alpha < 0$ (including the case of the harmonic index, when $\alpha = -1$), Corollary 3.3 those of Du, Zhou and Trinajstić, and Corollary 3.2 the result of Tomescu and Kanwal (which holds for unicyclic graphs, when $m = n$). In section 2 we state some parametric inequalities which will be used in the last section. In section 3 we determine the connected graphs $G$ of order $n \geq 3$ with girth at least $k$ ($3 \leq k \leq n$) having minimum $\chi_\alpha(G)$ for $-1 \leq \alpha < 0$.  

2. Some preliminary results  

Let $g(n,k) = (n-k)(n-k+3)^\alpha + 2(n-k+4)^\alpha + (k-2)4^\alpha$. Note that $g(n,k) = \chi_\alpha(C_{k,n-k})$.  

Lemma 2.1. [8] The function $f(n,k) = k(k+3)^\alpha + 2(k+4)^\alpha + (n-k-2)4^\alpha$ is strictly decreasing in $k \geq 0$ for $-1 \leq \alpha < 0$.  

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Since \( g(n, k) = f(n, n - k) \) we deduce the following property.

**Corollary 2.1.** *The function \( g(n, k) \) is strictly increasing in \( k \), \( 3 \leq k \leq n \) for \( -1 \leq \alpha < 0 \).*

**Lemma 2.2.** [8] *The function*

\[
\psi(x) = 2(x + 5)^\alpha + (x - 1)(x + 4)^\alpha - x(x + 3)^\alpha
\]

*defined for \( x \geq 0 \) and \(-1 \leq \alpha < 0 \) is strictly decreasing.*

**Lemma 2.3.** [7] *Let \( uv \) be an edge of a graph \( G \) such that \( d(u) + d(v) \) is minimum. If \(-1 \leq \alpha < 0 \) then \( \chi_\alpha(G - uv) < \chi_\alpha(G) \).*

**Lemma 2.4.** [8] a) *Let \( x > 0 \). If \( \alpha < 0 \) or \( \alpha > 1 \) then \( (1 + x)^\alpha > 1 + \alpha x \).*

b) *Let \( x > 0 \). If \( \alpha < 0 \) or \( 1 < \alpha < 2 \) then \( (1 + x)^\alpha < 1 + \alpha x + \frac{\alpha(\alpha - 1)}{2} x^2 \) (for \( \alpha = 2 \) equality holds).*

**Lemma 2.5.** *The function \( g(n, k) \) is strictly subadditive in \( n \) for \(-1 \leq \alpha < 0 \), i.e.,*

\[
g(n_1 + n_2, k) < g(n_1, k) + g(n_2, k), \quad (1)
\]

*where \( n_1, n_2 \geq k \geq 3 \).*

**Proof.** *By letting \( n_1 + n_2 = n \geq 2k, n_1 = x \) we deduce \( n_2 = n - x \) and (1) leads to*

\[
g(x, k) + g(n - x, k) > g(n, k)
\]

*for every \( k \leq x \leq n - k \). Using formula for \( g(n, k) \) this inequality is equivalent to*

\[
(x - k)(x - k + 3)^\alpha + 2(x - k + 4)^\alpha + (n - x - k)(n - x - k + 3)^\alpha + 2(n - x - k + 4)^\alpha + (k - 2)4^\alpha
\]

\[
> (n - k)(n - k + 3)^\alpha + 2(n - k + 4)^\alpha. \quad (2)
\]

*Let*

\[
\eta(x) = (x - k)(x - k + 3)^\alpha + 2(x - k + 4)^\alpha + (n - x - k)(n - x - k + 3)^\alpha + 2(n - x - k + 4)^\alpha.
\]

*We have \( \eta(x) = \eta(n - x) \); we can write \( \eta(x) = \gamma(x) + \gamma(n - x) \), where*

\[
\gamma(x) = (x - k)(x - k + 3)^\alpha + 2(x - k + 4)^\alpha.
\]

*We get*

\[
\gamma''(x) = \alpha(\alpha - 1)(x - k)(x - k + 3)^{\alpha - 2} + 2\alpha(x - k + 3)^{\alpha - 1} + 2\alpha(\alpha - 1)(x - k + 4)^{\alpha - 2}
\]

\[
< \alpha(\alpha - 1)(x - k)(x - k + 3)^{\alpha - 2} + 2\alpha(x - k + 3)^{\alpha - 1} + 2\alpha(\alpha - 1)(x - k + 3)^{\alpha - 2}
\]

\[
= \alpha(x - k + 3)^{\alpha - 2}((\alpha + 1)(x - k + 2) + 2) < 0.
\]
Similarly, $\gamma''(n-x) < 0$, so $\eta''(x) < 0$, hence $\eta(x)$ is a concave function. Because $\eta(x) = \eta(n-x)$ where $k \leq x \leq n-k$, so the minimum of $\eta(x)$ is reached at $x = k$ and $x = n-k$. Replacing $x = k$ in (2) yields

$$k4^\alpha + (n - 2k)(n - 2k + 3)^\alpha + 2(n - 2k + 4)^\alpha > (n - k)(n - k + 3)^\alpha + 2(n - k + 4)^\alpha.$$ (3)

In order to prove (3) we shall consider a new variable $x = n \geq 2k$ and the function

$$\varphi(x) = (x - 2k)(x - 2k + 3)^\alpha + 2(x - 2k + 4)^\alpha - (x - k)(x - k + 3)^\alpha - 2(x - k + 4)^\alpha$$

defined for $x \geq 2k \geq 6$. We deduce

$$\varphi'(x) = (x - 2k + 3)^{\alpha-1}((x - 2k)(\alpha + 1) + 3) + 2\alpha(x - 2k + 4)^{\alpha-1}$$

$$- (x - k + 3)^{\alpha-1}((x - k)(\alpha + 1) + 3) - 2\alpha(x - k + 4)^{\alpha-1} > (x - 2k + 3)^{\alpha-1}(x(\alpha + 1)$$

$$- 2k(\alpha + 1) + 3 + 2\alpha) - (x - k + 3)^{\alpha-1}(x(\alpha + 1) - k(\alpha + 1) + 3) - 2\alpha(x - k + 4)^{\alpha-1}$$

$$= E(x, k, \alpha)(x - k + 4)^{\alpha-1}.$$

We have

$$E(x, k, \alpha) = \left[1 + \frac{k + 1}{x - 2k + 3}\right]^{1-\alpha} [x(\alpha + 1) - 2k(\alpha + 1) + 3 + 2\alpha]$$

$$- \left[1 + \frac{1}{x - k + 3}\right]^{1-\alpha} [x(\alpha + 1) - k(\alpha + 1) + 3] - 2\alpha.$$

By Lemma 2.5 we get

$$E(x, k, \alpha) > \left[1 + \frac{(1 - \alpha)(k + 1)}{x - 2k + 3}\right] [x(\alpha + 1) - 2k(\alpha + 1) + 3 + 2\alpha]$$

$$- \left[1 + \frac{1 - \alpha}{x - k + 3} + \frac{\alpha(\alpha - 1)}{2(x - k + 3)^2}\right] [x(\alpha + 1) - k(\alpha + 1) + 3] - 2\alpha$$

$$= -\alpha k(1 + \alpha) + \alpha(\alpha - 1)F(x, k, \alpha),$$

where

$$F(x, k, \alpha) = \frac{(1 + \alpha)(k - x) - 3}{2(x - k + 3)^2} - \frac{3}{x - k + 3} + \frac{k + 1}{x - 2k + 3}.$$

Finally,

$$F(x, k, \alpha) > \frac{k - x - 3}{(x - k + 3)^2} - \frac{3}{x - k + 3} + \frac{k + 1}{x - 2k + 3} = -\frac{4}{x - k + 3} + \frac{k + 1}{x - 2k + 3} > 0$$

since $k \geq 3$ implies $\frac{k + 1}{x - 2k + 3} > \frac{1}{x - k + 3}$.

Because $\varphi'(x) > 0$ it follows that $\varphi(x)$ is strictly increasing and (3) holds if it holds for $n = 2k$ and $k \geq 3$. Substituting $n = 2k$ in (3) yields

$$(k + 2)4^\alpha > k(k + 3)^\alpha + 2(k + 4)^\alpha,$$

which is true because $k \geq 3$. \qed
Lemma 2.6. Let \( G \in \mathcal{G}_n \) such that \( g(G) \geq k \). We have \( \Delta(G) \leq n - k + 2 \) and the bound is tight.

Proof. Let \( v \in V(G) \) such that \( d(v) = \Delta(G) \). Suppose that \( v \) belongs to a cycle in \( G \) and denote by \( C \) a shortest cycle containing \( v \). It follows that \( v \) is adjacent to exactly 2 vertices of \( C \), thus implying \( \Delta(G) \leq n - l + 2 \), where \( l \) denotes the length of \( C \). Since \( l \geq g(G) \) we obtain \( \Delta(G) \leq n - g(G) + 2 \leq n - k + 2 \).

If \( v \) does not belong to any cycle in \( G \), it follows that a shortest cycle of \( G \) contains at most one vertex in the set \( N(v) \) and we deduce \( \Delta(G) + 1 + g(G) - 1 \leq n \), or \( \Delta(G) \leq n - g(G) < n - k + 2 \). The bound is reached because \( \Delta(C_{k,n-k}) = n - k + 2 \).

3. Main Results

Theorem 3.1. Let \( G \) be a connected graph of order \( n \geq 3 \) and size \( m \geq n \) with girth \( g(G) \geq k \) \((3 \leq k \leq n) \). If \(-1 \leq \alpha < 0\) then \( \chi_\alpha(G) \geq g(n, k) = (n-k)(n-k+3)^\alpha + 2(n-k+4)^\alpha + (k-2)^4\alpha \).

Equality holds if and only if \( G = C_{k,n-k} \).

Proof. The proof is by induction on \( m + n \). For \( n = 3 \) we have \( m = k = 3 \), \( G = C_3 \) and in this case the property holds. Also we can suppose that \( n \geq k + 1 \), since for \( n = k \) there exists a unique graph, namely \( C_{n,0} = C_n \). Let \( m \geq n \geq 4 \). Suppose the property is true for smaller values of \( m + n \). Let \( G \in \mathcal{G}_n \) having girth \( g(G) \geq k \) such that \( \chi_\alpha(G) \) is minimum. We shall consider two cases: A. \( \delta(G) = 1 \) and B. \( \delta(G) \geq 2 \).

A. In this case there exists a pendant vertex \( u \in V(G) \) and let \( uv \in E(G) \). We have \( d(v) = d \geq 2 \) and let \( N(v) \setminus \{u\} = \{u_1, \ldots, u_{d-1}\} \). Since \( G \) is a connected graph containing at least one cycle, we get that there exists at least one vertex in \( \{u_1, \ldots, u_{d-1}\} \) with degree at least 2. Suppose there exists exactly one vertex in this set with degree at least 2, say \( w \). Let \( d(w) = s \geq 2 \) and let \( N(w) \setminus \{v\} = \{v_1, \ldots, v_{s-1}\} \). Define \( G_1 = G \setminus \{vw_1, \ldots, vv_{s-1}\} \). It follows that \( G_1 \in \mathcal{G}_{n-1} \) and \( g(G_1) = g(G) \geq k \). We deduce

\[
\chi_\alpha(G) - \chi_\alpha(G_1) = (d-1)[(d+1)^\alpha - (d+s)^\alpha] + \sum_{i=1}^{s-1}[(d(v_i) + s)^\alpha - (d(v_i) + d + s - 1)^\alpha] > 0
\]

since \( d \geq 2 \) and \( s \geq 2 \). This contradicts the assumption about the minimality of \( G \).

So we deduce that there exist at least two vertices in \( \{u_1, \ldots, u_{d-1}\} \) with degree at least 2, thus implying \( d \geq 3 \). Let \( G_2 = G \setminus \{u\} \). We have \( G_2 \in \mathcal{G}_{n-1} \) and \( g(G_2) = g(G) \geq k \). It follows that

\[
\chi_\alpha(G) = \chi_\alpha(G_2) + (d+1)^\alpha + \sum_{i=1}^{d-1}[(d + d(u_i))\alpha - (d + d(u_i) - 1)^\alpha].
\]

Since the function \( h(x) = (d + x)^\alpha - (d + x - 1)^\alpha \) has \( h'(x) > 0 \) for any \( \alpha < 0 \), one has

\[
\sum_{i=1}^{d-1}[(d + d(u_i))\alpha - (d + d(u_i) - 1)^\alpha] \geq 2[(d + 2)^\alpha - (d + 1)^\alpha] + (d - 3)[(d + 1)^\alpha - d\alpha],
\]
Corollary 3.1. equality holds if and only if two degrees of \(u_1, \ldots, u_{d-1}\) are equal to 2, the remaining ones being 1.

By the induction hypothesis we obtain \(\chi_\alpha(G_2) \geq g(n - 1, k)\), which yields

\[\chi_\alpha(G) \geq g(n - 1, k) + 2(d + 2)^\alpha + (d - 4)(d + 1)^\alpha - (d - 3)d^\alpha.\]

Inequality \(g(n - 1, k) + 2(d + 2)^\alpha + (d - 4)(d + 1)^\alpha - (d - 3)d^\alpha \geq g(n, k)\) is equivalent to

\[(n - k - 1)(n - k + 2)^\alpha + 2(d + 2)^\alpha + (d - 4)(d + 1)^\alpha - (d - 3)d^\alpha \geq (n - k - 2)(n - k + 3)^\alpha + 2(n - k + 4)^\alpha. \quad (4)\]

Let \(g(x) = 2(x + 2)^\alpha + (x - 4)(x + 1)^\alpha - (x - 3)x^\alpha\). Since \(g(x) = \psi(x - 3)\), by Lemma 2.3 it follows that \(g(x)\) is strictly decreasing for \(x \geq 3\) and \(-1 \leq \alpha < 0\). Note that by Lemma 2.7 we have \(d \leq \Delta(G) \leq n - k + 2\) since \(g(G) \geq k\). This leads to the inequality \(2(d + 2)^\alpha + (d - 4)(d + 1)^\alpha - (d - 3)d^\alpha \geq 2(n - k + 4)^\alpha + (n - k - 2)(n - k + 3)^\alpha - (n - k - 1)(n - k + 2)^\alpha\) and equality holds only for \(d = n - k + 2\). In this case (4) becomes an equality. Summarizing, we have \(\chi_\alpha(G) = g(n, k)\) only if \(G_2 = C_{k,n-1-k}\), \(d(v) = n - k + 2\) and \(v\) is adjacent in \(G_2\) to \(k - 1\) pendant vertices and to \(2\) vertices of degree \(2\). We have \(\chi_\alpha(G) \geq g(n, k)\) and equality holds only if \(G = C_{k,n-k}\).

B. In this case \(\delta(G) \geq 2\). We shall prove that \(\chi_\alpha(G) \geq g(n, k)\). Since \(\delta(G) \geq 2\) we may assume that \(m \geq n + 1\) because \(m = n\) implies \(G\) is 2-regular, hence \(G = C_n = C_{n,0}\) and \(\chi_\alpha(C_n) = g(n, n) \geq g(n, k)\) for every \(3 \leq k \leq n - 1\) by Corollary 2.2.

Let \(e = uv \in E(G)\) such that \(d(u) + d(v)\) is minimum. By Lemma 2.4 we have \(\chi_\alpha(G - uv) < \chi_\alpha(G)\). Since \(m \geq n + 1\), \(g(G - uv) \geq k\) holds since the cyclomatic number of \(G\) is equal to two. We shall consider two subcases B1 and B2, according to \(e\) is a cut-edge in \(G\) or not, respectively.

B1. \(e\) being a cut-edge, \(G - e\) has two components, say \(G_1\) and \(G_2\), where \(u \in V(G_1)\) and \(v \in V(G_2)\). By denoting \(|V(G_i)| = n_i\) for \(1 \leq i \leq 2\) we get \(n = n_1 + n_2\). Because \(\delta(G) \geq 2\) and \(g(G) \geq k\) we obtain that each \(G_i\) has at least one cycle and \(g(G_i) \geq g(G) \geq k\), which implies \(n_i \geq k\) for \(1 \leq i \leq 2\). By induction, since \(G_i \in \mathcal{G}_{n_i}\) for each \(i\), we deduce \(\chi_\alpha(G) > \chi_\alpha(G - e) = \chi_\alpha(G_1) + \chi_\alpha(G_2) \geq g(n_1, k) + g(n_2, k) > g(n, k)\) by Lemma 2.6.

B2. In this case \(G - e\) is a connected graph of order \(n\) and size \(m - 1\), with \(m - 1 \geq n\) and \(g(G - e) \geq k\). By induction \(\chi_\alpha(G - e) \geq g(n, k)\), which implies \(\chi_\alpha(G) > g(n, k)\) and the proof is complete.

Since extremal graph \(C_{k,n-k}\) has girth equal to \(k\), we deduce the following corollary.

**Corollary 3.1.** Let \(G\) be a connected graph of order \(n \geq 3\) and size \(m \geq n\) with girth \(g(G) = k\) \((3 \leq k \leq n)\). If \(-1 \leq \alpha < 0\) then \(\chi_\alpha(G) \geq g(n, k)\). Equality holds if and only if \(G = C_{k,n-k}\).

Since \(H(G) = 2\chi_{-1}(G)\), the result also holds for the harmonic index.

If \(-1 \leq \alpha < 0\) note that \(C_{k,n-k}\) is not extremal for zeroth-order general Randić index \(0R_\alpha(G)\).

If \(G_1\) denotes the graph consisting of \(C_{n-2}\) and two pendant edges incident to two distinct vertices of \(C_{n-2}\), then we get \(0R_\alpha(G_1) < 0R_\alpha(C_{n-2,2})\). This inequality is equivalent to \(2 \cdot 3^\alpha < 2^\alpha + 4^\alpha\),

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which is valid by Jensen’s inequality.

Because by Corollary 2.2 the minimum of the function $g(n, k)$ is reached only for $k = 3$, an extremal property deduced by other means for unicyclic graphs in [2] follows:

**Corollary 3.2.** If $-1 \leq \alpha < 0$, in the class of connected graphs $G$ of fixed order $n$ and variable size $m \geq n$, $\chi_\alpha(G)$ is minimum if and only if $G = C_{3,n-3}$.

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**References**


