Reciprocal complementary distance spectra and reciprocal complementary distance energy of line graphs of regular graphs

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Abstract
The reciprocal complementary distance (RCD) matrix of a graph $G$ is defined as $RCD(G) = [r_{ij}]$ where $r_{ij} = \frac{1}{1+D-d_{ij}}$ if $i \neq j$ and $r_{ij} = 0$, otherwise, where $D$ is the diameter of $G$ and $d_{ij}$ is the distance between the vertices $v_i$ and $v_j$ in $G$. The $RCD$-energy of $G$ is defined as the sum of the absolute values of the eigenvalues of $RCD(G)$. Two graphs are said to be $RCD$-equienergetic if they have same $RCD$-energy. In this paper we show that the line graph of certain regular graphs has exactly one positive $RCD$-eigenvalue. Further we show that $RCD$-energy of line graph of these regular graphs is solely depends on the order and regularity of $G$. This results enables to construct pairs of $RCD$-equienergetic graphs of same order and having different $RCD$-eigenvalues.

Keywords: Reciprocal complementary distance eigenvalues, adjacency eigenvalues, line graphs, reciprocal complementary distance energy
Mathematics Subject Classification : 05C50, 05C12
DOI: 10.5614/ejgta.2015.3.2.10

1. Introduction
Molecular matrices, encoding in various ways the topological information, are an important source of structural descriptors for quantitative structure property relationships (QSPR) and quantitative structure activity relationships (QSAR) models [6]. A large number of molecular matrices

Received: 01 June 2015, Revised: 08 September 2015, Accepted: 13 October 2015.
were defined in the chemical literature. One of these is reciprocal complementary distance (RCD) matrix.

Let \( G \) be a simple, undirected, connected graph with \( n \) vertices and \( m \) edges. Let the vertices of \( G \) be labeled as \( v_1, v_2, \ldots, v_n \). The adjacency matrix of a graph \( G \) is the square matrix \( A = A(G) = [a_{ij}] \), in which \( a_{ij} = 1 \) if \( v_i \) is adjacent to \( v_j \) and \( a_{ij} = 0 \), otherwise. The eigenvalues of the adjacency matrix \( A(G) \) are the adjacency eigenvalues of \( G \), and these will be labeled as \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \) and their collection is called as an adjacency spectra of \( G \) [3].

The distance between the vertices \( v_i \) and \( v_j \), denoted by \( d_{ij} \), is the length of the shortest path between them. The diameter of a graph \( G \), denoted by \( \text{diam}(G) \), is the maximum distance between any pair of vertices of \( G \). A graph \( G \) is said to be \( r \)-regular graph if all of its vertices have same degree equal to \( r \).

The reciprocal complementary distance (RCD) between the vertices \( v_i \) and \( v_j \), denoted by \( rc_{ij} \) is defined as \( rc_{ij} = \frac{1}{1+D-d_{ij}} \), where \( D \) is the diameter of \( G \) and \( d_{ij} \) is the distance between \( v_i \) and \( v_j \) in \( G \).

The reciprocal complementary distance matrix [6, 7] of a graph \( G \) is an \( n \times n \) real symmetric matrix \( RCD(G) = [rc_{ij}] \), where

\[
rc_{ij} = \begin{cases} 
\frac{1}{1+D-d_{ij}}, & \text{if } i \neq j \\
0, & \text{otherwise}.
\end{cases}
\]

The eigenvalues of \( RCD(G) \) labeled as \( \mu_1 \geq \mu_2 \geq \cdots \geq \mu_n \) are said to be the RCD-eigenvalues of \( G \) and their collection is called RCD-spectra of \( G \). Two non-isomorphic graphs are said to be RCD-cospectral if they have same RCD-spectra.

The reciprocal complementary distance energy (RCD-energy) of a graph \( G \) is defined as

\[
RCD E(G) = \sum_{i=1}^{n} |\mu_i| .
\] (1)

The Eq. (1) is defined in full analogy with the ordinary graph energy \( E(G) \), defined as [4]

\[
E(G) = \sum_{i=1}^{n} |\lambda_i| .
\] (2)

Two graphs \( G_1 \) and \( G_2 \) are said to be equienergetic if \( E(G_1) = E(G_2) \) [1, 2, 8, 11, 12, 16]. For more details on \( E(G) \) one can refer [8].

Two connected graphs \( G_1 \) and \( G_2 \) are said to be reciprocal complementary distance equienergetic or RCD-equiequie energetic if \( RCD E(G_1) = RCD E(G_2) \). Of course, RCD-cospectral graphs are RCD-equiequie energetic. In this paper we obtain the RCD-eigenvalues and RCD-energy of line
graphs of certain regular graphs. Further we show that the $RCD$-energy of line graphs of certain regular graphs is solely depends on the order and regularity of a graph. Thus infinitely many pairs of $RCD$-equienergetic graphs can be constructed such that they have equal number of vertices, equal number of edges and are non $RCD$-cospectral.

We need following results.

**Theorem 1.1.** [3] If $G$ is an $r$-regular graph, then its maximum adjacency eigenvalue is equal to $r$.

**Theorem 1.2.** [13] Let $G$ be an $r$-regular graph of order $n$. If $r, \lambda_2, \ldots, \lambda_n$ are the adjacency eigenvalues of $G$, then the adjacency eigenvalues of $\overline{G}$, the complement of $G$, are $n - r - 1$ and $-\lambda_i - 1$, $i = 2, 3, \ldots, n$.

The line graph of $G$, denoted by $L(G)$ is the graph whose vertices corresponds to the edges of $G$ and two vertices of $L(G)$ are adjacent if and only if the corresponding edges are adjacent in $G$ [5]. If $G$ is a regular graph of order $n$ and of degree $r$ then the line graph $L(G)$ is a regular graph of order $nr/2$ and of degree $2r - 2$.

**Theorem 1.3.** [14] If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the adjacency eigenvalues of a regular graph $G$ of order $n$ and of degree $r$, then the adjacency eigenvalues of $L(G)$ are

$$
\lambda_i + r - 2, \quad i = 1, 2, \ldots, n, \quad \text{and} \quad -2, \quad n(r - 2)/2 \text{ times}.
$$

![Figure 1: The forbidden induced subgraphs](image)

**Theorem 1.4.** [9, 10] For a connected graph $G$, $\text{diam}(L(G)) \leq 2$ if and only if none of the three graphs $F_1$, $F_2$ and $F_3$ of Fig. 1 is an induced subgraph of $G$.

**Lemma 1.1.** [15] If for any two adjacent vertices $u$ and $v$ of a graph $G$, there exists a third vertex $w$ which is not adjacent to any of $u$ and $v$, then

(i) $\overline{G}$ is connected and

(ii) $\text{diam} \ (\overline{G}) \leq 2$. 


2. **RCD-eigenvalues**

**Theorem 2.1.** Let $G$ be an $r$-regular graph on $n$ vertices and diam$(G) = 2$. If $r, \lambda_2, \ldots, \lambda_n$ are the adjacency eigenvalues of $G$, then its RCD-eigenvalues are $n - 1 - \frac{r}{2}$ and $-1 - \frac{\lambda_i}{2}$, $i = 2, 3, \ldots, n$.

**Proof.** Since $G$ is an $r$-regular graph, $1 = [1,1,\ldots,1]'$ is an eigenvector of $A = A(G)$ corresponding to the eigenvalue $r$. Set $z = \frac{1}{\sqrt{n}}1$ and let $P$ be an orthogonal matrix with its first column equal to $z$ such that $P'AP = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$. Since diam$(G) = 2$, RCD$(G)$ can be written as $\text{RCD}(G) = J - I - \left(\frac{1}{2}\right)A$, where $J$ is the matrix whose all entries are equal to 1 and $I$ is an identity matrix. It follows that

$$P'(\text{RCD})P = P'(J - I - \frac{1}{2}A)P = P'JP - I - \frac{1}{2}P'AP,$$

where we have used the fact that any column of $P$ other than the first column is orthogonal to the first column. Hence the eigenvalues of RCD$(G)$ are $n - 1 - \frac{r}{2}$ and $-1 - \frac{\lambda_i}{2}$, $i = 2, 3, \ldots, n$.

**Theorem 2.2.** If $G$ is an $r$-regular, connected graph of order $n \geq 4$ and if none of the three graphs $F_1$, $F_2$ and $F_3$ of Fig. 1 is an induced subgraph of $G$, then $L(G)$ has exactly one positive RCD-eigenvalue, equal to $r(n - 2)/2$.

**Proof.** Let $r, \lambda_2, \lambda_3, \ldots, \lambda_n$ be the adjacency eigenvalues of a regular graph $G$. Then from Theorem 1.3, the adjacency eigenvalues of $L(G)$ are

$$\lambda_i + r - 2, \quad i = 1, 2, \ldots, n,$$

and

$$-2, \quad n(r - 2)/2 \text{ times}. \quad (3)$$

The graph $G$ is regular of degree $r$ and has order $n$. Therefore $L(G)$ is a regular graph on $nr/2$ vertices and of degree $2r - 2$. As none of the three graphs $F_1$, $F_2$ and $F_3$ of Fig. 1 is an induced subgraph of $G$, from Theorem 1.4, diam$(L(G)) = 2$. Therefore from Theorem 2.1 and Eq. (3), the RCD-eigenvalues of $L(G)$ are

$$r(n - 2)/2, \quad \text{and}$$

$$-(\lambda_i + r)/2, \quad i = 2, 3, \ldots, n \quad \text{and}$$

$$0, \quad n(r - 2)/2 \text{ times}. \quad (4)$$

All adjacency eigenvalues of a regular graph of degree $r$ satisfy the condition $-r \leq \lambda_i \leq r$ [3]. Therefore $\lambda_i + r \geq 0$, $i = 1, 2, \ldots, n$. The theorem follows from Eq. (4).
3. RCD-energy

**Theorem 3.1.** If $G$ is an $r$-regular, connected graph of order $n \geq 4$ and if none of the three graphs $F_1$, $F_2$ and $F_3$ of Fig. 1 is an induced subgraph of $G$, then

$$RCDE(L(G)) = r(n - 2).$$

**Proof.** Bearing in mind Theorem 2.2 and Eq. (4), the RCD-energy of $L(G)$ is computed as:

$$RCDE(L(G)) = \frac{r(n - 2)}{2} + \sum_{i=2}^{n} \frac{(\lambda_i + r)}{2} + |0| \times \frac{n(r - 2)}{2}$$

$$= r(n - 2) \quad \text{since} \quad \sum_{i=2}^{n} \lambda_i = -r.$$

From Theorem 3.1, we see that the RCD-energy of the line graph of a regular graph $G$, that does not contain $F_i$, $i = 1, 2, 3$, as an induced subgraph is fully determined by the order $n$ and degree $r$ of $G$.

Let $K_n$ be the complete graph on $n$ vertices, $K_{k,k}$ be the complete bipartite graph on $2k$ vertices and $CP(k)$ be the cocktail party graph (a regular graph on $n = 2k$ vertices and of degree $2k - 2$) [3]. None of the three graphs $F_1$, $F_2$ and $F_3$ of Fig.1 is an induced subgraph of these graphs. Therefore from Theorem 3.1 we have following:

**Corollary 3.1.**

(i) $RCDE(L(K_n)) = n^2 - 3n + 2$, for $n \geq 4$.

(ii) $RCDE(L(K_{k,k})) = 2k(k - 1)$, for $k \geq 2$.

(iii) $RCDE(L(CP(k))) = 4(k - 1)^2$, for $k \geq 2$.

**Theorem 3.2.** Let $G$ be an $r$-regular graph of order $n$. Let $L(G)$ be the line graph of $G$ such that for any two adjacent vertices $u$ and $v$ of $L(G)$, there exists a third vertex $w$ in $L(G)$ which is not adjacent to any of $u$ and $v$.

(i) If the smallest adjacency eigenvalue of $G$ is greater than or equal to $3 - r$, then

$$RCDE\left(\overline{L(G)}\right) = \frac{3n(r - 2)}{2}.$$

(ii) If the second largest adjacency eigenvalue of $G$ is at most $3 - r$, then

$$RCDE\left(\overline{L(G)}\right) = \left(\frac{nr}{2}\right) + 2r - 3.$$

**Proof.** Let the adjacency eigenvalues of $G$ be $r, \lambda_2, \ldots, \lambda_n$. From Theorem 1.3, the adjacency eigenvalues of $L(G)$ are

$$2r - 2, \quad \text{and}$$

$$\lambda_i + r - 2, \quad i = 2, 3, \ldots, n, \quad \text{and}$$

$$-2, \quad n(r - 2)/2 \text{ times.} \quad (5)$$
From Theorem 1.2 and the Eq. (5), the adjacency eigenvalues of $L(G)$ are
\[
\begin{align*}
(nr/2) - 2r + 1, \quad \text{and} \\
-\lambda_i - r + 1, \quad i = 2, 3, \ldots, n, \quad \text{and} \\
1, \quad n(r - 2)/2 \times.
\end{align*}
\]
\[\text{(6)}\]

Since for any two adjacent vertices $u$ and $v$ of $L(G)$ there exists a third vertex $w$ which is not adjacent to any of $u$ and $v$ in $L(G)$, by Lemma 1.1, $\text{diam}(L(G)) = 2$. Therefore by Theorem 2.1 and Eq. (6), the RCD-eigenvalues of $L(G)$ are
\[
\begin{align*}
(nr/4) + r - (3/2), \quad \text{and} \\
\frac{\lambda_i + r - 3}{2}, \quad i = 2, 3, \ldots, n, \quad \text{and} \\
(-3/2), \quad n(r - 2)/2 \times.
\end{align*}
\]
\[\text{(7)}\]
Therefore
\[
RCD_{DE}(L(G)) = \left| \frac{nr}{4} + r - \frac{3}{2} + \sum_{i=2}^{n} \frac{\lambda_i + r - 3}{2} \right| + \left| -\frac{3}{2} \right| n(r - 2)/2. \tag{8}
\]

(i) By assumption, $\lambda_i + r - 3 \geq 0$, $i = 2, 3, \ldots, n$, then from Eq. (8)
\[
RCD_{DE}(L(G)) = \frac{nr}{4} + r - \frac{3}{2} + \frac{1}{2} \sum_{i=2}^{n} \lambda_i + (n - 1) \left( r - \frac{3}{2} \right) + \frac{3n(r - 2)}{4}
\]
\[
= \frac{3n(r - 2)}{2} \quad \text{since} \quad \sum_{i=2}^{n} \lambda_i = -r.
\]

(ii) By assumption, $\lambda_i + r - 3 < 0$, $i = 2, 3, \ldots, n$, then from Eq. (8)
\[
RCD_{DE}(L(G)) = \frac{nr}{4} + r - \frac{3}{2} - \frac{1}{2} \sum_{i=2}^{n} \lambda_i - (n - 1) \left( r - \frac{3}{2} \right) + \frac{3n(r - 2)}{4}
\]
\[
= \frac{nr}{2} + 2r - 3 \quad \text{since} \quad \sum_{i=2}^{n} \lambda_i = -r.
\]
Some of the examples of \( r \)-regular graphs whose second largest adjacency eigenvalue is at most \( 3 - r \) and the diameter of the complement of their line graph is equal to two are a 5-vertex cycle \( C_5 \), a 5-vertex complete graph \( K_5 \), a 6-vertex cycle \( C_6 \) and a complete bipartite graph \( K_{3,3} \).

**Corollary 3.2.** Let \( G \) be a cubic graph of order \( n \). Let \( L(G) \) be the line graph of \( G \) such that for any two adjacent vertices \( u \) and \( v \) of \( L(G) \), there exists a third vertex \( w \) in \( L(G) \) which is not adjacent to any of \( u \) and \( v \). Then

\[
RCDE(L(G)) = \frac{3n + E(G)}{2}.
\]

**Proof.** Substituting \( r = 3 \) in Eq. (8) we get

\[
RCDE(L(G)) = \left| \frac{3n}{4} + \frac{3}{2} \right| + \sum_{i=2}^{n} \left| \frac{\lambda_i}{2} \right| + \left| \frac{3}{2} \right| = \frac{3n}{4} + \frac{3}{2} + \frac{1}{2}(E(G) - 3) + \frac{3n}{4} = \frac{3n + E(G)}{2}.
\]

4. **RCD-equieenergetic graphs**

**Lemma 4.1.** Let \( G_1 \) and \( G_2 \) be regular graphs of the same order and of the same degree. Then following holds:

(i) \( L(G_1) \) and \( L(G_2) \) are of the same order, same degree and have the same number of edges.
(ii) \( L(G_1) \) and \( L(G_2) \) are of the same order, same degree and have the same number of edges.

**Proof.** Statement (i) follows from the fact that the line graph of a regular graph is a regular and that the number of edges of \( G \) is equal to the number of vertices of \( L(G) \). Statement (ii) follows from the fact that the complement of a regular graph is a regular and that the number of vertices of a graph and its complement is equal.

**Lemma 4.2.** Let \( G_1 \) and \( G_2 \) be regular, connected graphs of the same order \( n \geq 4 \) and of the same degree. Let none of the three graphs \( F_1, F_2 \) and \( F_3 \) of Fig. 1 be an induced subgraph of \( G_i \), \( i = 1, 2 \). Then \( L(G_1) \) and \( L(G_2) \) are RCD-cospectral if and only if \( G_1 \) and \( G_2 \) are cospectral.

**Proof.** Follows from Eqs. (3) and (4).

**Lemma 4.3.** Let \( G_1 \) and \( G_2 \) be regular graphs of the same order and of the same degree. Let for \( i = 1, 2 \), \( L(G_i) \) be the line graph of \( G_i \) such that for any two adjacent vertices \( u_i \) and \( v_i \) of \( L(G_i) \), there exists a third vertex \( w_i \) in \( L(G_i) \) which is not adjacent to any of \( u_i \) and \( v_i \). Then \( L(G_1) \) and \( L(G_2) \) are RCD-cospectral if and only if \( G_1 \) and \( G_2 \) are cospectral.
Proof. Follows from Eqs. (5), (6) and (7).

**Theorem 4.1.** Let $G_1$ and $G_2$ be regular, connected, non cospectral graphs of the same order $n \geq 4$ and of the same degree $r$. Let none of the three graphs $F_1$, $F_2$ and $F_3$ of Fig. 1 be an induced subgraph of $G_i$, $i = 1, 2$. Then line graphs $L(G_1)$ and $L(G_2)$ form a pair of non RCD-cospectral, RCD-equienergetic graphs of equal order and of equal number of edges.

Proof. Follows from Lemma 4.1, Lemma 4.2 and Theorem 3.1.

**Theorem 4.2.** Let $G_1$ and $G_2$ be regular, non cospectral graphs of the same order and of the same degree $r$. Let for $i = 1, 2$, $L(G_i)$ be the line graph of $G_i$ such that for any two adjacent vertices $u_i$ and $v_i$ of $L(G_i)$, there exists a third vertex $w_i$ in $L(G_i)$ which is not adjacent to any of $u_i$ and $v_i$.

(i) If the smallest adjacency eigenvalue of $G_i$, $i = 1, 2$ is greater than or equal to $3 - r$, then $L(G_1)$ and $L(G_2)$ form a pair of non RCD-cospectral, RCD-equienergetic graphs of equal order and of equal number of edges.

(ii) If the second largest adjacency eigenvalue of $G_i$, $i = 1, 2$ is at most $3 - r$, then $L(G_1)$ and $L(G_2)$ form a pair of non RCD-cospectral, RCD-equienergetic graphs of equal order and of equal number of edges.

Proof. Follows from Lemma 4.1, Lemma 4.3 and Theorem 3.2.

**Theorem 4.3.** Let $G_1$ and $G_2$ be non cospectral, cubic equienergetic graphs of the same order. Let for $i = 1, 2$, $L(G_i)$ be the line graph of $G_i$ such that for any two adjacent vertices $u_i$ and $v_i$ of $L(G_i)$, there exists a third vertex $w_i$ in $L(G_i)$ which is not adjacent to any of $u_i$ and $v_i$. Then $L(G_1)$ and $L(G_2)$ form a pair of non RCD-cospectral, RCD-equienergetic graphs of equal order and of equal number of edges.

Proof. Follows from Lemma 4.1, Lemma 4.3 and Corollary 3.2.

**Acknowledgement**

Authors are thankful to anonymous referee for his/her valuable suggestions. The first author H. S. Ramane is thankful to the University Grants Commission (UGC), Govt. of India for support through research grant under UPE FAR-II grant No. F 14-3/2012 (NS/PE). Another author A. S. Yalnaik is thankful to the University Grants Commission (UGC), Govt. of India for support through Rajiv Gandhi National Fellowship No. F1-17.1/2014-15/RGNF-2014-15-SC-KAR-74909.

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