Electronic Journal of Graph Theory and Applications **5** (1) (2017), 94–111

Electronic Journal of Graph Theory and Applications

Some new graceful generalized classes of diameter six trees

Debdas Mishra[a], Sushant Kumar Rout[b], Purna Chandra Nayak[c]

[a] *C. V. Raman College of Engineering, Bhubaneswar, India*
[b] *College of Engineering and Technology, Bhubaneswar, India*
[c] *Bhadrak Autonomous College, Bhadrak, India*

debdasmishra@gmail.com, sushant9027@gmail.com, debmaths@yahoo.com

Abstract

Here we denote a *diameter six tree* by $(c; a_1, a_2, \ldots, a_m; b_1, b_2, \ldots, b_n; c_1, c_2, \ldots, c_r)$, where c is the center of the tree; $a_i, i = 1, 2, \ldots, m$, $b_j, j = 1, 2, \ldots, n$, and $c_k, k = 1, 2, \ldots, r$ are the vertices of the tree adjacent to c; each a_i is the center of a diameter four tree, each b_j is the center of a star, and each c_k is a pendant vertex. Here we give graceful labelings to some new classes of diameter six trees $(c; a_1, a_2, \ldots, a_m; b_1, b_2, \ldots, b_n; c_1, c_2, \ldots, c_r)$ in which a diameter four tree may contain any combination of branches with the total number of branches odd though with some conditions on the number of odd, even, and pendant branches. Here by a branch we mean a star, i.e. we call a star an odd branch if its center has an odd degree, an even branch if its center has an even degree, and a pendant branch if it is a pendant vertex.

Keywords: graceful labeling, diameter six tree, odd and even branches, component moving transformation
Mathematics Subject Classification : 05C78
DOI:10.5614/ejgta.2017.5.1.10

1. Introduction

Definition 1.1. *[13] A* ***graceful labeling*** *of a (p, q) graph G is an injection $f : V(G) \to \{0, 1, 2, \ldots, q\}$ such that, when each edge uv of G is assigned the label $|f(u) - f(v)|$, the resulting edge labels (or weights) are distinct from the set $\{1, 2, 3, \ldots, q\}$. A graph that admits a graceful labeling is said to be graceful. As for a tree $q = p - 1$, f is also onto and hence bijective.*

Received: 8 September 2014, Revised: 14 January 2017, Accepted: 26 January 2017.

Definition 1.2. *A **diameter six tree** is a tree which has a representation of the form* $(c; a_1, a_2, \ldots, a_m; b_1, b_2, \ldots, b_n; c_1, c_2, \ldots, c_r)$*, where* c *is the center of the tree;* $a_i, i = 1, 2, \ldots, m$, $b_j, j = 1, 2, \ldots, n$, *and* $c_k, k = 1, 2, \ldots, r$ *are the vertices of the tree adjacent to* c*; each* a_i *is the center of a diameter four tree, each* b_j *is the center of a star, and each* c_k *is a pendant vertex. We observe that in a diameter six tree with above representation* $m \geq 2$*, i.e. there should be at least two (vertices)* a_i*'s adjacent to* c *which are the centers of diameter four trees. Here we use the notation* D_6 *to denote a diameter six tree. A combination of branches incident on any* $a_i,\ 0 \leq i \leq m$*, can be represented by a triple* (x, y, z)*, where* $x,\ y$*, and* z *represent the number of odd, even, and pendant branches, respectively, incident on* a_i*. Here we use the symbols* e *and* o *to represent a non-zero even number and an odd number, respectively. For example:* $(e, 0, o)$ *means an even number of odd branches, no even branch, and an odd number of pendant branches. If in a triple* e *or* o *appears more than once then it does not mean that the corresponding branches are equal in number, for example,* (e, e, o) *does not mean that the number of odd branches is equal to the number of even branches.*

In the literature [3, 4, 5, 6, 12] we find that all trees up to diameter five are graceful. As far as diameter six trees are concerned, only *banana trees* are graceful [1, 2, 3, 4, 6, 7, 8, 14, 15, 12, 16]. From literature [2] a *banana tree* is a tree obtained by connecting a vertex v to one leaf of each of any number of stars (v is not in any of the stars). Chen et al. [2] conjectured that banana trees are graceful. Bhat-Nayak and Deshmukh [1], Murugan and Arumugam [8] and Vilfred [16] gave graceful labelings to different classes of banana trees. Sethuraman and Jesintha [6, 7, 14, 15]) proved that all banana trees and extended banana trees (graphs obtained by joining a vertex to one leaf of each of any number of stars by a path of length of at least two) are graceful. In this paper we give graceful labelings to some new classes of diameter six trees $(a_1, a_2, \ldots, a_m; b_1, b_2, \ldots, b_n; c_1, c_2, \ldots, c_r)$ with each $a_i, i = 1, 2, \ldots, m_1, m_1 \leq m$, is attached to $(o, 0, 0)$. In the diameter six trees with the above representation a diameter four tree may contain any combination of branches with the total number of branches odd though with some conditions on the number of odd, even, and pendant branches.

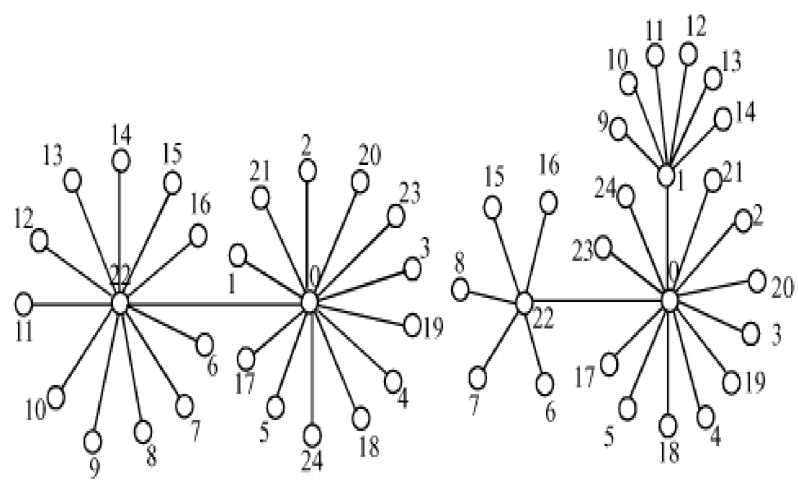
2. Preliminaries

Now we state some existing terminologies results borrowed from [5, 9, 10, 11] to prove our main result.

Definition 2.1. *For an edge* $e = \{u, v\}$ *of a tree* T*, we define* $u(T)$ *as that connected component of* $T - e$ *which contains the vertex* u*. Here we say* $u(T)$ *is a component incident on the vertex* v*. If* a *and* b *are vertices of a tree* T*,* $u(T)$ *is a component incident on* a*, and* $b \notin u(T)$ *then deleting the edge* $\{a, u\}$ *from* T *and making* b *and* u *adjacent is termed **as the component*** $u(T)$ ***has been transferred or moved from*** a ***to*** b*. In this paper by the label of the component "*$u(T)$*" we mean the label of the vertex* u*. Let* T *be a tree and* a *and* b *be two vertices of* T*. By* $a \to b$ ***transfer*** *we mean that some components from* a *have been moved to* b*. If we consider successive transfers* $a_1 \to a_2$*,* $a_2 \to a_3$*,* $a_3 \to a_4$*,* $\ldots$ *we simply write* $a_1 \to a_2 \to a_3 \to a_4 \ldots$ *transfer. In the transfer* $a_1 \to a_2 \to \ldots \to a_{n-1} \to a_n$*, each vertex* $a_i, i = 1, 2, \ldots, n-1$ *is called a vertex of transfer.*

Lemma 2.1. *[5] Let f be a graceful labeling of a tree T; let a and b be two vertices of T; let $u(T)$ and $v(T)$ be two components incident on a where $b \notin u(T) \cup v(T)$. Then the following hold:*
(i) if $f(u)+f(v) = f(a)+f(b)$ then the tree T^* obtained from T by moving the components $u(T)$ and $v(T)$ from a to b is also graceful.
(ii) if $2f(u) = f(a)+f(b)$ then the tree T^{**} obtained from T by moving the component $u(T)$ from a to b is also graceful.

Definition 2.2. *Let T be a labelled tree with a labeling f. We consider the vertices of T whose labels form the sequence $(a, b, a-1, b+1, a-2, b+2)$ (respectively, $(a, b, a+1, b-1, a+2, b-2)$). Let a be adjacent to some vertices having labels different from the above labels. The $a \longrightarrow b$ transfer is called* ***a transfer of the first type*** *if the labels of the transferred components constitute a set of consecutive integers. The $a \longrightarrow b$ transfer is called* ***a transfer of the second type*** *if the labels of the transferred components can be divided into two segments, where each segment is a set of consecutive integers. A sequence of eight transfers of the first type $a \to b \to a-1 \to b+1 \to a \to b \to a-1 \to b+1 \to a-2$ (respectively, $a \to b \to a+1 \to b-1 \to a \to b \to a+1 \to b-1 \to a+2$), is called a* ***backward double 8 transfer of the first type*** *or* ***BD8TF*** *a to $a-2$ (respectively, a to $a+2$). A sequence of five transfers of the first type $a \to b+1 \to a-1 \to b \to a-2 \to b+2$ (respectively, $a \to b-1 \to a+1 \to b \to a+2 \to b-2$), is called* ***a 5 - transfer of the first type*** *or in brief* ***5TF*** *a to $b+2$ (respectively, a to $b-2$). A sequence of four transfers of the first type $a \to b+1 \to a-1 \to b+1 \to a-2$ (respectively, $a \to b-1 \to a+1 \to b-1 \to a+2$), is called* ***a 1 - jump transfer of the first type*** *or in brief* ***1JTF*** *a to $a-2$ (respectively, a to $a+2$). A sequence of two transfers of the first type $a \to b+1 \to a-2$ (respectively, $a \to b-1 \to a+2$), is called* ***a 2 - jump transfer of the first type*** *or in brief* ***2JTF*** *a to $a-2$ (respectively, a to $a+2$). A sequence of two transfers of the first type $a \to b+2 \to a-3$ (respectively, $a \to b-2 \to a+3$), is called* ***a 4 - jump transfer of the first type*** *or in brief* ***4JTF*** *a to $a-3$ (respectively, a to $a+3$).*



(a) (b)

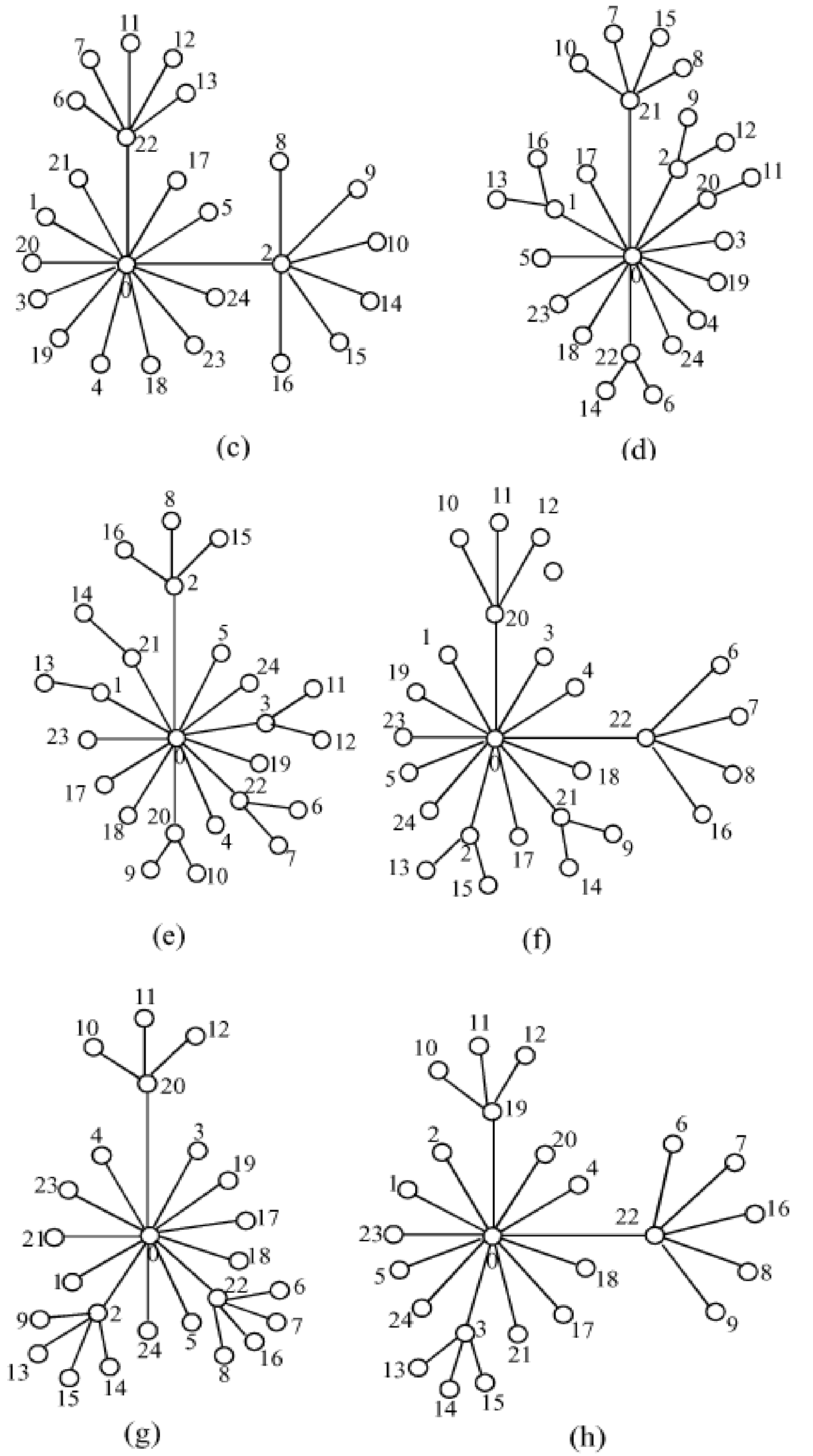


Figure 1. The graceful trees in (b), (c), d), (e), (f), (g), and (h) are obtained from the graceful tree in (a) by applying transfers of the first type $22 \rightarrow 1$, the transfer of second type $22 \rightarrow 2$, BD8TF 22 to 20, 5TF 22 to 3, 1JTF 22 to 20, 2JTF 22 to 20, and 4JTF 22 to 18, respectively.

Theorem 2.1. *[9, 10, 11] In a graceful labeling f of a graceful tree T, let a and b be the labels of two vertices. Let a be attached to a set A of vertices (or components) having labels $n, n+1, n+2, \ldots, n+p$ (different from the above vertex labels), which satisfy $(n+1+i)+(n+p-i) = a+b, i \geq 0$ (respectively, $(n+i)+(n+p-1-i) = a+b, i \geq 0$). Then the following hold.*

- *(a) By making a transfer $a \to b$ of first type we can keep an odd number of components at a from the set A and move the rest to b, and the resultant tree thus formed will be graceful.*
- *(b) If A contains an even number of elements, then by making a sequence of transfers of the second type $a \to b \to a-1 \to b+1 \to a-2 \to b+2 \to \ldots$ (respectively, $a \to b \to a+1 \to b-1 \to a+2 \to b-2 \to \ldots$), an even number of elements from A can be kept at each vertex of the transfer, and the resultant tree thus formed is graceful.*
- *(c) By a BD8TF a to $b+1$ (respectively, $b-1$), we can keep an even number of elements from A at $a, b, a-1$, and $b+1$ (respectively, $a, b, a+1$, and $b-1$), and move the rest to $a-2$ (respectively, $a+2$). By a 5TF a to $a-2$ (respectively, $a+2$), we can keep an even number of components at a and $a-2$ (respectively, a and $a+2$) and an odd number of components at the remaining vertices of the transfer and move the rest to $b+2$ (respectively, $b-2$). By a 1JTF a to $b+1$ (respectively, $b-1$), we can keep an even number of elements from A at a, $a-1$, and $b+1$ (respectively, a, $a+1$, and $b-1$) and no component at b, and move the rest to $a-2$ (respectively, $a+2$). By a 2JTF a to $b+1$ (respectively, $b-1$), we can keep an even number of components at a and $b+1$ (respectively, $b-1$) and no component at b and $a-1$ (respectively, $a+1$), and move the rest to $a-2$ (respectively, $a+2$). By making a 4JTF a to $b+2$ (respectively, $b-2$), we can keep an odd number (≥ 3) of components at a and $b+2$ (respectively, $b-2$) and no component at b, $a-1$, $b+1$, and $a-2$ (respectively, b, $a+1$, $b-1$, and $a+2$), and move the rest to $a-3$ (respectively, $a+3$). The resultant tree formed in each of the above cases is graceful.*
- *(d) Consider the transfer $R : a \to b \to a-1 \to b+1 \to \ldots \to z$ (respectively, $a \to b \to a+1 \to b-1 \to \ldots \to z$), with $z = a-p_1$ or $b+p_2$ (respectively, $a+r_1$ or $b-r_2$), such that R is partitioned as $R : T_1 \to T_2 \to T_3 \to \ldots \to T_n$, where each $T_i, 1 \leq i \leq n$, is either a transfer of the first type or any of the derived transfers. Construct a tree T^* from T by making the transfer R part wise, i.e. first the transfer T_1, then T_2 and so on. The tree T^* is graceful.*
- *(e) Consider the transfer $R' : a \to b \to a-1 \to b+1 \to \ldots \to \ldots$ (respectively, $a \to b \to a+1 \to b-1 \to \ldots \to \ldots$), such that R' is partitioned as $R' : T'_1 \to T'_2$, where T'_1 is sequence of transfers consisting of the transfers of the first type and the derived transfers and T'_2 is a sequence of transfer of the second type. The tree T^{**} obtained from T by making the transfer R' is graceful.*

Lemma 2.2. *[5] If g is a graceful labeling of a tree T with n edges then the labeling g_n defined as $g_n(x) = n - g(x)$, for all $x \in V(T)$, called the inverse transformation of g is also a graceful labeling of T.*

3. Results

Construction 3.1. *We construct a diameter six tree $D_6 = (c; a_1, a_2, \ldots, a_m; b_1, b_2, \ldots, b_n; c_1, c_2, \ldots, c_r)$ with degree of each a_i and b_j an odd number. Suppose that o_i, e_i, and p_i are the number of odd, even, and pendant branches incident on the center $a_i, 1 \leq j \leq m$.*
(1) The vertex a_i may be attached to one of the following combinations with the conditions specified.
(a) $(o, 0, 0)$.
(b) (o, e, e) *or* $(o, e, 0)$ *with either* $e_i - p_i \equiv 0 \ (mod\ 4)$ *or* $o_i \geq 3$.
(c) (o, o, o) *with* $e_i \geq 3$ *and either* $e_i - p_i \equiv 2 \ (mod\ 4)$ *or* $o_i \geq 3$ *and* $e_i - p_i \geq 4$.
(d) $(o, 0, e)$ *or* (o, e, e) *with* $e_i \equiv 0 (mod\ 4)$, $p_i \equiv 0 \ (mod\ 4)$, $o_i \geq \frac{p_i}{2} + 2$, *and at least* $\frac{p_i}{2}$ *odd branches contain 3 or more pendant vertices.*
(2) The combinations of branches incident on a_i and a_{i+1} may be one of the following.
(a) The vertex a_i is attached to (e, e, o) or $(0, e, o)$ (respectively, (e, o, e)) or $(0, o, e)$ and a_{i+1} is attached to (e, o, e) or $(0, o, e)$ (respectively, (e, e, o))or $(0, e, o)$ with the conditions $a_i \geq p_i + 1$, $a_{i+1} \geq p_{i+1} + 1$, and $[e_i + e_{i+1} - p_i - p_{i+1}] \equiv 2 \ (mod\ 4)$.
(b) Both the vertices a_i and a_{i+1} are attached to (e, e, o) or $(0, e, o)$ with the conditions $e_i \geq p_i + 1$, $e_{i+1} \geq p_{i+1} + 1$, $[e_i + e_{i+1} - p_i - p_{i+1}] \equiv 0 \ (mod\ 4)$, and $[e_i + e_{i+1} - p_i - p_{i+1}] \geq 4$.
(c) Both the vertices a_i and a_{i+1} are attached to (e, o, e) or $(0, o, e)$ with the conditions $e_i \geq p_i - 1$, $e_{i+1} \geq p_{i+1} - 1$, $[e_i + e_{i+1} - p_i - p_{i+1}] \equiv 0 \ (mod\ 4)$, and $[e_i + e_{i+1} - p_i - p_{i+1}] \geq 0$.
(d) The vertices a_i and a_{i+1} are attached to either $(e, o, 0)$ or $(0, o, 0)$ with $[e_i + e_{i+1}] \equiv 0 \ (mod\ 4)$.
(e) The vertex a_i is attached to either $(0, o, 0)$ or $(e, o, 0)$ (respectively, $(0, o, e)$ or (e, o, e)) and the vertex a_{i+1} is attached to either $(0, o, e)$ or (e, o, e) (respectively, $(0, o, 0)$ or $(e, o, 0)$) with $[e_i + e_{i+1} - p_i - p_{i+1}] \equiv 0 \ (mod\ 4)$ and $[e_i + e_{i+1} - p_i - p_{i+1}] \geq 4$.
(f) The vertex a_i is attached to either $(0, o, 0)$ or $(e, o, 0)$ (respectively, $(0, e, o)$ or (e, e, o)) and the vertex a_{i+1} is attached to either $(0, e, o)$ or (e, e, o) (respectively, $(0, o, 0)$ or $(e, o, 0)$) with $e_i \geq p_i + 3$, $e_{i+1} \geq p_{i+1} + 3$, and $[e_i + e_{i+1} - p_i - p_{i+1}] \equiv 2 \ (mod\ 4)$
(g) The vertices a_i and a_{i+1} are attached to either $(e, 0, o)$ or (e, e, o) with $p_i + p_{i+1} \equiv 0 \ (mod\ 4)$, $e_i \equiv 0 \ (mod\ 4)$, $e_{i+1} \equiv 0 \ (mod\ 4)$, $o_r \geq n_r + 2, r = i, i+1$, and at least n_r odd branches incident on a_r contain 3 or more pendant vertices, where $n_r = \frac{p_r + 1}{2}$ if $p_r \equiv 1 \ (mod\ 4)$ and $\frac{p_r - 1}{2}$ if $p_r \equiv 3 \ (mod\ 4)$.
(h) Both the vertices a_i and a_{i+1} are attached to (e, o, e) with $e_i + e_{i+1} \equiv 0 \ (mod\ 4)$, $o_r \geq \frac{p_r}{2}, r = i, i+1$, and at least $\frac{p_r}{2}$ odd branches incident on a_r contain 3 or more pendant vertices.
(i) The vertex a_i is attached to $(o, e, 0)$ (respectively, (o, e, e)) and the vertex a_{i+1} is attached to (o, e, e) (respectively, $(o, e, 0)$) with $[e_i + e_{i+1} - p_i - p_{i+1}] \equiv 0 \ (mod\ 4)$ and $[e_i + e_{i+1} - p_i - p_{i+1}] \geq 4$.
(j) The vertex a_i is attached to $(o, e, 0)$ or (o, e, e) (respectively, (o, o, o)) and the vertex a_{i+1} is attached to (o, o, o) (respectively, $(o, e, 0)$) or (o, e, e) with $[e_i + e_{i+1} - p_i - p_{i+1}] \equiv 2 \ (mod\ 4)$ and $[e_i + e_{i+1} - p_i - p_{i+1}] \geq 2$.
(k) Both the vertices a_i and a_{i+1} are attached to (o, o, o) or (o, e, e) with one of the following conditions:
(I) $[e_r - p_r] \equiv 0 \ (mod\ 4)$, *and* $[e_r - p_r] \geq 0$, *for* $r = i, i+1$.
(II) $[e_r - p_r] \equiv 2 \ (mod\ 4)$, *and* $[e_r - p_r] \geq 2$, *for* $r = i, i+1$.

(III) $[e_i - p_i] \equiv 0\ (mod\ 4)$, $[e_{i+1} - p_{i+1}] \equiv 2\ (mod\ 4)$, $[e_i - p_i] \geq 0$, $[e_{i+1} - p_{i+1}] \geq 2$, $o_{i+1} \geq 3$.
(IV) $[e_i - p_i] \equiv 2\ (mod\ 4)$, $[e_{i+1} - p_{i+1}] \equiv 0\ (mod\ 4)$, $[e_i - p_i] \geq 2$, $[e_{i+1} - p_{i+1}] \geq 0$, $o_i \geq 3$.
(l) Both the vertices a_i and a_{i+1} are attached to (o, e, e) with $e_i + e_{i+1} \equiv 0\ (mod\ 4)$, and for $r = i, i+1$, $p_r \equiv 0\ (mod\ 4)$, $o_r \geq \frac{p_r}{2} + 1$ and at least $\frac{p_r}{2}$ odd branches incident on a_r contains 3 or more pendant vertices. ◇

Example 3.1. The diameter six tree $D_6\ (c; a_1, a_2, a_3, a_4, a_5, a_6; b_1, b_2, b_3; c_1, c_2, c_3)$ in Figure 2 is of the type in Construction 3.1. Here $o_1 = 3, e_1 = 0, p_1 = 0$; $o_2 = 2, e_2 = 2, p_2 = 1$; $o_3 = 0, e_3 = 3, p_3 = 2$; $o_4 = 1, e_4 = 1, p_4 = 1$; $o_5 = 1, e_5 = 1, p_5 = 1$; $o_6 = 3, e_6 = 4, p_6 = 4$; Thus, a_1 is attached to $(o, 0, 0)$, a_2 is attached to (e, e, o), a_3 is attached to $(0, o, e)$, a_4 is attached to (o, o, o), a_5 is attached to (o, o, o), and a_6 is attached to (o, e, e).

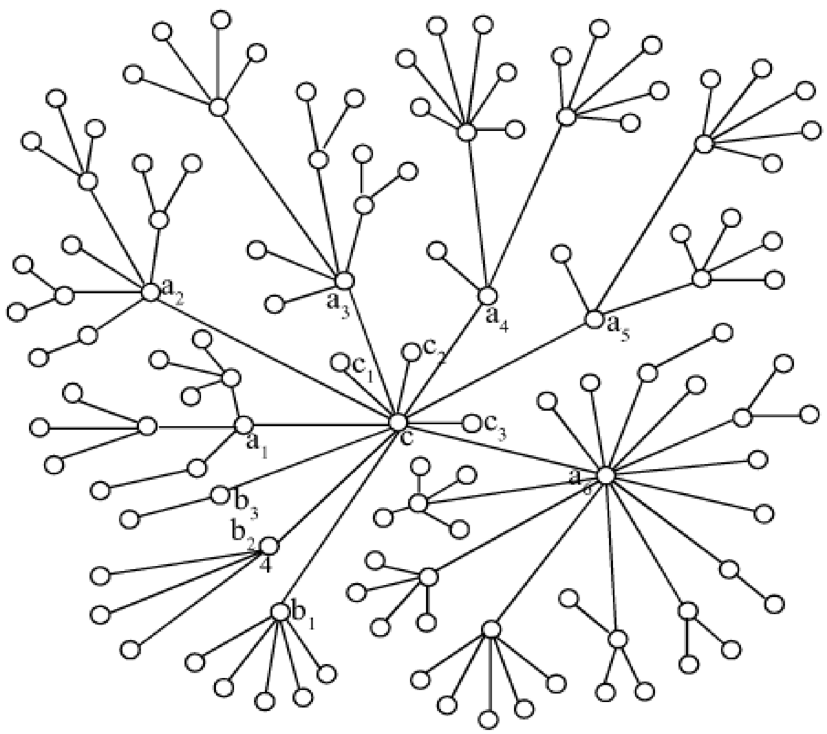


Figure 2. A diameter six tree of the type in Construction 3.1

Theorem 3.1. *The diameter six tree D_6 in Construction 3.1 is graceful.*

Proof.
Case - I. Let $m + n$ be odd. Let $|E(D_6)| = q$ and $deg(a_0) = m + n = 2k + 1$. Suppose that for $i = 1, 2, \ldots, m$, $o_i + e_i + p_i = deg(a_i) - 1 = 2\lambda_i + 1$. We proceed as per the following steps to get a graceful labeling of D_6.
1. Remove the pendant vertices adjacent to c and represent the new graceful tree by $D_6^{(1)}$. Consider the graceful tree G as represented in Figure 3.

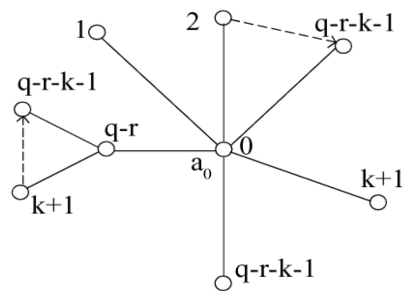


Figure 3. The graceful tree G.

2. Let $A = \{k+1, k+2, \ldots, q-k-r-1\}$. Observe that $(k+i)+(q-r-k-i) = q-r$. Designate the vertices adjacent to a_0, i.e. $a_i, i = 1, 2, \ldots, m$, $b_j, j = 1, 2, \ldots, n$ as:

$$a_i = \begin{cases} q - r - \frac{i-1}{2} & \text{if } i \text{ is odd} \\ \frac{i}{2} & \text{if } i \text{ is even} \end{cases} \quad \text{and } b_j = \begin{cases} \begin{cases} q - r - \frac{m+j-1}{2} & \text{if } j \text{ is odd} \\ \frac{m+j}{2} & \text{if } j \text{ is even} \end{cases} & \text{if } m \text{ is even} \\ \\ \begin{cases} \frac{m+j}{2} & \text{if } j \text{ is odd} \\ q - r - \frac{m+j-1}{2} & \text{if } j \text{ is even} \end{cases} & \text{if } m \text{ is odd} \end{cases}$$

Let A be the set of all pendant vertices adjacent to $a_1 = q - r$ in G. The set A can be written as $A = \{z_1, z_2, \ldots, z_s\}$, where for $1 \le i \le s = 2\sum_{i=1}^{m}(2\lambda_i + 1)$,

$$z_i = \begin{cases} q - r - k - \frac{i}{2} & \text{if } i \text{ is even} \\ k + \frac{i+1}{2} & \text{if } i \text{ is odd} \end{cases}$$

3. Consider the sequence of transfer $T_1 : a_1 \to a_2 \to a_3 \to \ldots \to a_m \to b_1 \to \ldots \to b_n \to z_1$, i.e. $q-r \to 1 \to q-r-1 \to 2 \to \ldots \to k \to q-r-k+1 \to k \to q-r-k \to k+1$ with each transfer is a transfer of first type of the vertex "labels in set A". Observe that the transfer T_1 and the set A satisfy the hypothesis of Theorem 2.1. Carry out the transfer T_1 and keep $2\lambda_i + 1$ elements of A at the vertices a_i and a desired odd number of vertices at each vertex b_j. Let A_1 be the set of vertices of A that have come to the vertex $k+1$. Let the resultant graceful tree thus formed be G_1.
4. Consider the transfer $T_2 : z_1 \to z_2 \to z_3 \to z_4 \ldots \to z_s$, where $s = \sum_{i=1}^{m}[2\lambda_j + 1]$.

The manner in which we have moved the vertices of A in step 3, we notice that the first $2\lambda_1 + 1$ vertices in T_2 are incident on a_1, the next $2\lambda_2 + 1$ vertices in T_2 are incident on a_2, and so on. Further, we observe that the set A_1 and the vertices z_1 and z_2 satisfy the hypothesis of Lemma 2.1. We partition the transfer $T_2 : T_2^{(1)} \to T_2^{(2)} \to T_2^{(3)} \to \ldots \to T_2^{(m)}$ and carry out the transfer T_3 by successively carrying out the transfers $T_2^{(1)}, T_2^{(2)}, \ldots, T_2^{(m)}$ in order. Each transfer $T_2^{(i)}, i = 1, 2, \ldots, m$ consists of sequence of transfers of the first type and one or more of the derived transfers. Here the transfer $T_2^{(i)} : a_{s_{i-1}+1} \to a_{s_{i-1}+2} \to \ldots \to a_{s_i} \to a_{s_i+1}$, where for $i = 1, 2, \ldots, m$, $s_i = \sum_{j=1}^{i}(2\lambda_j + 1)$ and the vertices $a_{s_{i-1}+1}, a_{s_{i-1}+2}, \ldots, a_{s_i}$ are incident on the path a_i.

We start with the transfer $T_2^{(1)}$ or $T_2^{(1)} \to T_2^{(2)}$ for the cases (1) and (2), respectively.

Case (1): Let a_1 be attached to one of the combinations in (1). Here we carry out the transfer $T_2^{(1)} : z_1 \to z_2 \to \ldots \to z_{2\lambda_1+1}$.
Case (a): Here $T_2^{(1)}$ consists of $2\lambda_1 + 1$ successive transfers of the first kind.
Case (b): If $e_1 - p_1 \equiv 0 \ (mod\ 4)$, then $T_2^{(1)}$ consists of $\frac{e_1}{4}$ successive BD8TF followed by the o_1 successive transfers of the first kind. If $e_1 - p_1 \equiv 2 \ (mod\ 4)$ then $o_1 \geq 3$ and as such $T_2^{(1)}$ consists of one 5TF, followed by $\frac{e_1-2}{4}$ successive BD8TF, and finally $o_1 - 3$ successive transfers of the first kind.
Case (c): If $e_1 - p_1 \equiv 2(mod\ 4)$ then $T_2^{(1)}$ consists of o_1 successive transfers of the first type, followed by $\frac{p_1}{2}$ successive 2JTF, followed by $\frac{e_1-p_1-2}{4}$ successive BD8TF, finally one 1JTF. If $e_1 - p_1 \equiv 0(mod\ 4)$ then $o_1 \geq 3$, and $T_2^{(1)}$ consists of one 5TF, followed by $o_1 - 3$ successive transfers of the first type, followed by $\frac{p_1}{2}$ successive 2JTF, followed by $\frac{e_1-p_1-4}{4}$ successive BD8TF, finally one 1JTF.
Case (d): In this case $T_3^{(1)}$ consists of $\frac{e_1}{4}$ successive BD8TF, followed by $\frac{p_1}{4}$ successive 4JTF, and finally $o_1 - \frac{p_1}{2}$ successive transfers of the first kind. **Case (2):** Suppose a_1 and a_2 are attached to one of the combinations in (2). Here we carry out the transfer $T_2^{(1)} \to T_2^{(2)} : z_1 \to z_2 \to \ldots \to z_{2(\lambda_1+\lambda_2+1)}$.
Case (a): Let $[(e_1 + e_2) - (p_1 + p_2)] = 4l_1 + 2$. Here $T_2^{(1)} \to T_2^{(2)}$ consists of o_1 successive transfers of the first type, followed by $[\frac{p_1-1}{2}]$ successive 2JTF, followed by one 1JTF, followed by l_1 successive BD8TF, followed by $[\frac{p_2}{2}]$ successive 2JTF, and finally o_2 successive transfers of the first type.
Case (b): Let $[(e_1 + e_2) - (p_1 + p_2)] = 4l_2, l_2 \geq 1$. Here $T_2^{(1)} \to T_2^{(2)}$ consists of o_1 successive transfers of the first type, followed by one 1JTF, followed by $\frac{p_1-1}{2}$ successive 2JTF, followed by $(l_2 - 1)$ successive BD8TF, followed by $\frac{p_2-1}{2}$ successive 2JTF, followed by one 1JTF, and finally o_2 successive transfers of the first type.
Case (c): In this case $T_2^{(1)}$ consists of o_1 successive transfers of the first type, followed by $\frac{e_1-p_1}{4}$ successive BD8TFs followed by, $\frac{p_1+p_2}{2}$ successive 2JTF, followed by $\frac{e_2-p_2}{4}$, and finally, o_2 successive transfers of the first type.
Case (d): Here$T_2^{(1)}$ consists of o_1 successive transfers of the first type, followed by $\frac{e_1+e_2}{4}$ successive BD8TF, and finally o_2 successive transfers of the first type.
Case (e)-(i): Let $[(e_1+e_2)-(p_1+p_2)] = 4l_3, l_3 \geq 1$. Here $T_2^{(1)}$ consists of o_1 successive transfers of the first type, followed by $\frac{p_1}{2}$ successive 2JTF, followed by $\frac{l_3}{4}$ successive BD8TF, followed by $\frac{p_2}{2}$ successive 2JTF, and finally o_2 successive transfers of the first type.
Case (f): Let $[(e_1+e_2)-(p_1+p_2)] = 4l_4+2, l_4 \geq 1$. Here $T_2^{(1)}$ consists of o_1 successive transfers of the first type, followed by l_4 successive BD8TF, followed by one 1JTF, followed by $\frac{p_2}{2}$ successive 2JTF, and finally o_2 successive transfers of the first type (respectively, o_1 successive transfers of the first type, followed by $\frac{p_1}{2}$ successive 2JTF, followed by one 1JTF, followed by l_4 successive BD8TF, and finally o_2 successive transfers of the first type).
Case (g): Here $T_2^{(1)}$ consists of $\frac{e_1}{4}$ BD8TF, followed by $o_1 - n_1$ successive transfers of the first type, followed by $\frac{p_1+p_2}{4}$ successive 4JTF, followed by $o_2 - n_2$ successive transfers of the first type, and finally $\frac{e_2}{4}$ BD8TF.

Cases (h) : In this case $T_2^{(1)}$ consists of $o_1 - \frac{p_1}{2}$ successive transfers of the first type, followed by $\frac{p_1}{2}$ successive 4JTF, followed by $\frac{e_1+e_2}{4}$ successive BD8TF, followed by $\frac{p_2}{2}$ successive 4JTF, and finally o_2 successive transfers of the first type.

Case (j): Let $[(e_1 + e_2) - (p_1 + p_2)] = 4l_5 + 2, l_5 \geq 1$. In this case $T_2^{(1)}$ consists of o_1 successive transfers of the first type, followed by l_5 successive BD8TF, followed by $\frac{p_1}{2}$ successive 2JTF, followed by one 1JTF, followed by $\frac{p_2-1}{2}$ successive 2JTF, and finally, o_2 successive transfers of the first type (respectively, o_1 successive transfers of the first type, followed by one 1JTF, followed by $\frac{p_1-1}{2}$ successive 2JTF, followed by $\frac{p_1}{2}$ successive 2JTF, followed by l_5 successive BD8TF, and finally, o_2 successive transfers of the first type).

Case (k) - (I): In this case $T_2^{(1)}$ consists of o_1 successive transfers of the first type, followed by $\frac{e_1-p_1}{4}$ successive BD8TF, followed by $\frac{p_1+p_2}{4}$ successive 2JTF, followed by $\frac{e_2-p_2}{2}$ successive BD8TF, and finally o_2 successive transfers of the first type.

Case (k) - (II): In this case $T_2^{(1)}$ consists of o_1 successive transfers of the first type, followed by one 1JTF, followed by $\frac{e_1-p_1-2}{4}$ successive BD8TF, followed by $\frac{p_1+p_2-2}{2}$ successive 2JTF, followed by $\frac{e_2-p_2-2}{4}$ successive BD8TF, and finally o_2 successive transfers of the first type.

Case (k) - (III): In this case $T_2^{(1)}$ consists of o_1 successive transfers of the first type, followed by $\frac{e_1-p_1}{4}$ successive BD8TF, followed by $\frac{p_1+p_2}{4}$ successive 2JTF, followed by $\frac{e_2-p_2-2}{2}$ successive BD8TF, followed by one 5TF, and finally $o_2 - 3$ successive transfers of the first type.

Case (k) - (IV): In this case $T_2^{(1)}$ consists of $o_1 - 3$ successive transfers of the first type, followed by one 5TF, followed by $\frac{e_1-p_1-2}{4}$ successive BD8TF, followed by $\frac{p_1+p_2}{4}$ successive 2JTF, followed by $\frac{e_2-p_2}{2}$ successive BD8TF, and finally o_2 successive transfers of the first type.

Case (l): In this case $T_2^{(1)}$ consists of $o_1 - \frac{p_1}{2}$ successive transfers of the first type, followed by $\frac{p_1}{4}$ successive 4JTF, followed by $\frac{e_1+e_2}{4}$ successive BD8TF, followed by $\frac{p_2}{4}$ successive 4JTF, and finally $o_2 - \frac{p_2}{2}$ successive transfers of the first type.

In the similar manner we carry out the transfers $T_2^{(i)}$ successively in order by repeating the procedure in which we have accomplished the transfer $T_2^{(1)}$ and $T_{23}^{(1)} \to T_2^{(2)}$ respectively, for the Cases - (1) and (2) and complete the transfer $T_2 :\to T_2^{(1)} \to T_2^{(2)} \to T_2^{(3)} \to \ldots \to T_2^{(m)}$ so as to get back $D_6^{(1)}$ with a graceful labeling due to Theorem 2.1.

5. Now we attach r pendant vertices $c_1, c_2, \ldots, c_r$ to a_0 and assign them the labels $q - r + 1$, $q - r + 2, \ldots, q$, respectively, so as to from D_6 with a graceful labeling from the graceful tree $D_6^{(1)}$.

Example 3.2. The diameter six tree in Example 3.1 (Figure 2) is of the type in Theorem 3.1. Here $q = 118, m = 6, n = 4, r = 3$. The transfer $T_1 : a_1 \to a_2 \ldots \to b_n \to z_1$ in Step 3 is the transfer $118 \to 1 \to 117 \to \ldots \to 111 \to 5$. $T_2 : z_1 \to z_2 \to \ldots \to z_s$ in Step 4 is the transfer $5 \to 110 \to 6 \to \ldots \to 20 \to 95$ We first form the graceful tree G as in Figure 4. Figure 5 represents the graceful tree G_1 obtained after step 3. Figure 6 represents the graceful tree $D_6^{(1)}$ obtained after step 4. Figure 7 represents the given diameter six tree D_6 with a graceful labeling obtained by attaching the pendant vertices c_1, c_2, and c_3 assigning them the labels 116, 117, and 118 in step 5.

Case - II: Let $m + n$ be even. Then form a diameter six tree, say G_6 by removing the vertices $c_1, c_2, \ldots, c_r$, and b_n from D_6. Let $|E(G_6)| = q_1$. Give a graceful labeling to G_6 by following the

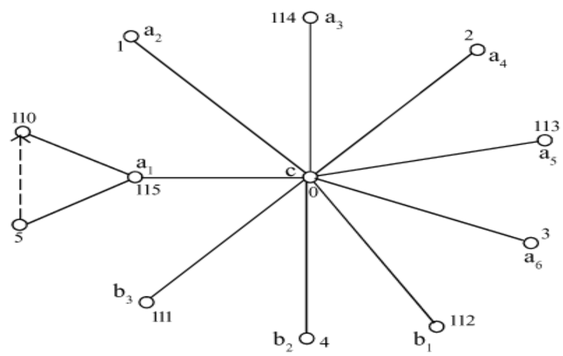


Figure 4. The tree G with a graceful labeling.

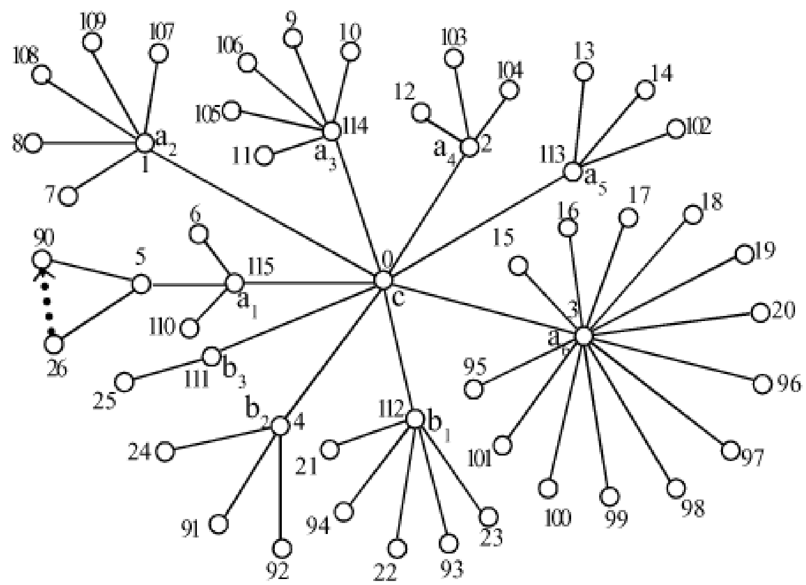


Figure 5. The graceful tree obtained after Step 3.

steps 1 to 9 involving giving a graceful labeling to $D_6^{(1)}$ in the proof for Case - I by replacing $q-r$ with q_1. Observe that in the graceful labeling of G_6, the vertex a_0 gets the label 0. Now attach the vertices c_1, c_2, $\ldots$, c_r, and b_n to a_0 and assign them the labels q_1+1, q_1+2, $\ldots$, q_1+r, and q_1+r+1, respectively. Obviously, the tree $G_6 \cup \{c_1, c_2, \ldots, c_r, b_n\}$ with the labelings mentioned above is graceful with a graceful labeling, say g. Then apply inverse transformation g_{q_1+r+1} to the above labeling of $G_6 \cup \{c_1, c_2, \ldots, c_r, b_n\}$. Now the vertex b_n gets the label 0. Let $deg(b_n) = p$. Finally, attach $p-1$ pendant vertices to b_n and assign them the labels q_1+r+2, q_1+r+3, $\ldots$, q_1+r+p, so as to get the tree D_6 with a graceful labeling. ■

The next result follows immediate from Theorem 3.1.

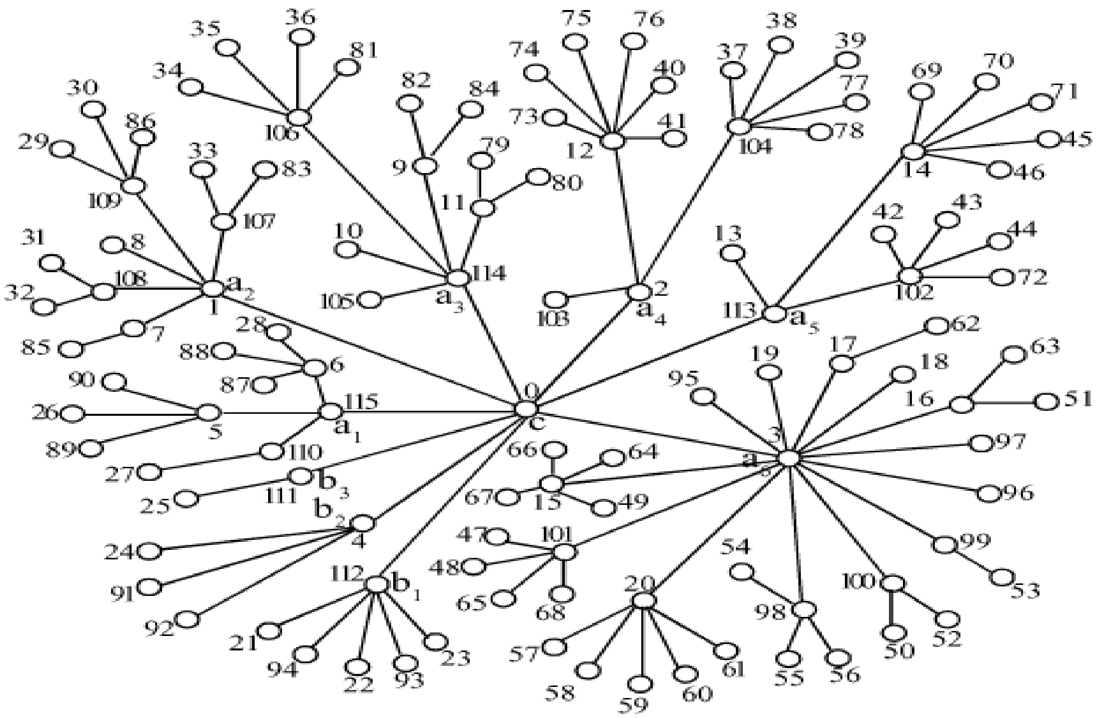


Figure 6. The graceful tree obtained after Step 4.

Construction 3.2. *If degrees of a_i and b_j are even, for $i = 1, 2, 3, \ldots, m; j = 1, 2, 3, \ldots, n$, and the centers $a_i, i = 1, 2, \ldots, m$, of diameter four trees are attached to combinations as in Theorem 3.1 then D_6 given by the following are graceful.*
(a): $D_6 = \{c; a_1, a_2, \ldots, a_m; b_1, b_2, \ldots, b_n\}$.
(b): $D_6 = \{c; a_1, a_2, \ldots, a_m; c_1, c_2, \ldots, c_r\}$ with m odd.
(c): $D_6 = \{c; a_1, a_2, \ldots, a_m\}$ with m odd.

Proof. Proofs of part (a) and (b) follow if we set $r = 0$ and $n = 0$, respectively in the proof involving Theorem 3.1. Proof of part (c) follows if we set $n = 0$ and $r = 0$ in the proof corresponding to Case - I of Theorem 3.1 . ■

Notation 3.1. Let $D_6 = \{a_0; a_1, a_2, \ldots, a_m; b_1, b_2, \ldots, b_n; c_1, c_2, \ldots, c_r\}$ be diameter six tree. We may have one of or both $n = 0$ and $r = 0$. For next couple of results we will consistent use the following notations.
n_e =Number of stars adjacent to a_0 with center having odd degree.
n_o = Number of stars adjacent to a_0 with center having even degree, i.e. $n = n_e + n_o$.

Theorem 3.2. *Let $m + n$ be odd, $n_e \equiv 0 \ mod\ 4$, degrees of a_i are even, for $i = 1, 2, 3, \ldots, m$. If the centers $a_i, i = 1, 2, \ldots, m$, of diameter four trees are attached to combinations as in Theorem 3.1 then*
(a) $D_6 = \{a_0; a_1, a_2, \ldots, a_m; b_1, b_2, \ldots, b_n; c_1, c_2, \ldots, c_r\}$ is graceful.
(b) $D_6 = \{a_0; a_1, \ldots, a_m; b_1, b_2, \ldots, b_n\}$ is graceful.

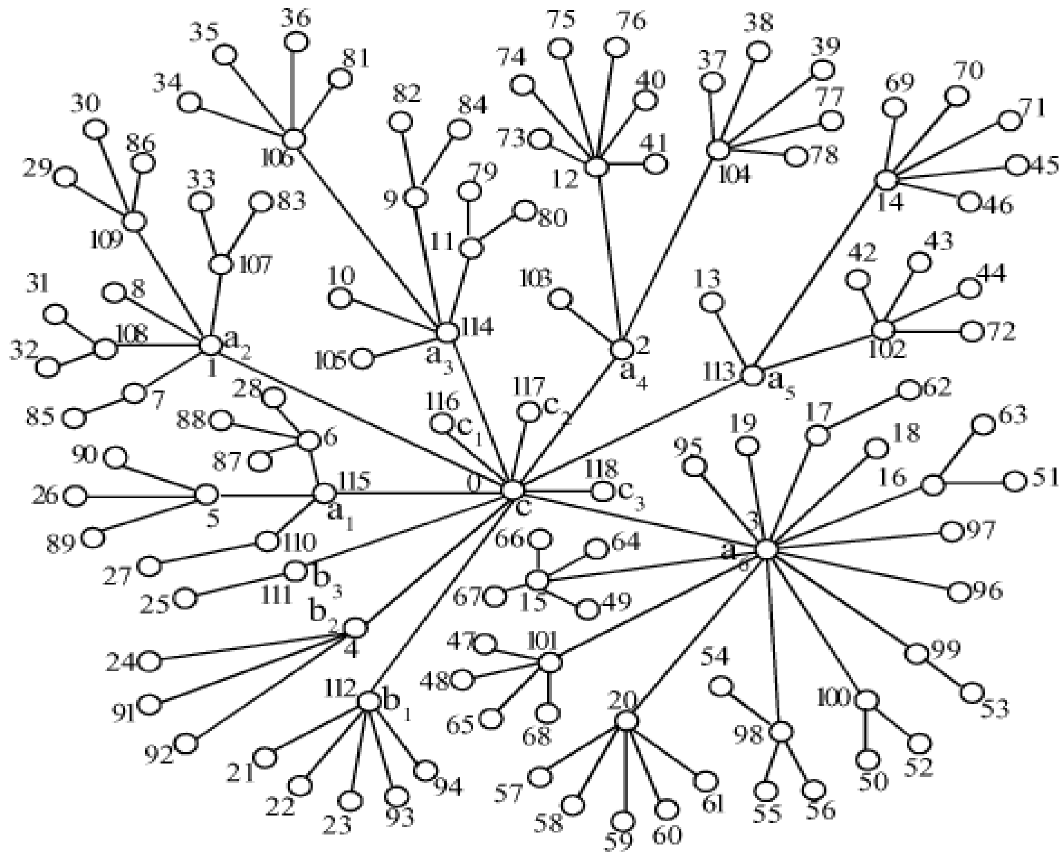


Figure 7. The graceful tree obtained after Step 5.

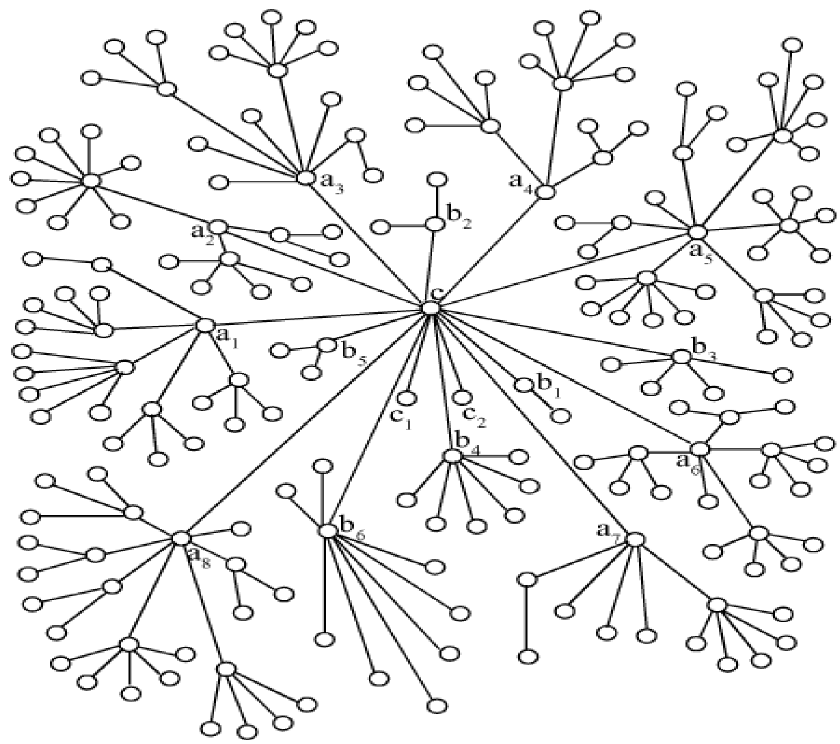
Proof. Let $|E(D_6)| = q$ and $deg(a_0) = m + n = 2k + 1$. Proceed as per the following steps. Let us first prove part (a). We repeat Steps 1 and 2 in the proof of Theorem 3.1 for Case -I. Consider the transfer $T_1 : a_1 \to a_2 \to a_3 \to \ldots \to a_m \to b_1 \to b_2 \to \ldots \to b_n \to z_1$ consisting of $m + n_o$ successive transfers of the first type, followed by $\frac{n_e}{4}$ successive BD8TF from vertex levels in the set A . Observe that the transfer T_1 and the set A satisfy the hypothesis of Theorem 2.1. Carry out the transfer T_1 keeping $2\lambda_i + 1$ elements of A at the vertices a_i, the desired odd number of vertices at $b_j, j = 1, 2, \ldots, n_o$, and the desired even number of vertices at $b_j, j = n_o + 1, n_o + 2, \ldots, n$ of T_1. By Theorem 2.1, the new tree, say G_1, thus formed is graceful. LetA_1 be the set of vertex labels of A which have come to the vertex z_1 after the transfer T_1. Finally, we repeat Steps 4 and 5 in the proof involving Theorem 3.1 for Case -I to get the tree D_6 with a graceful labeling. Proof of part (b) follows if we set $r = 0$ in the proof involving part (a).

Theorem 3.3. *Let $m + n$ be even, either $n_e \equiv 1 \ mod\ 4$ or $n_e \equiv 0 \ mod\ 4$ and $n_o \geq 1$, degrees of a_i are even, for $i = 1, 2, 3, \ldots, m$. If the centers $a_i, i = 1, 2, \ldots, m$, of diameter four trees are attached to combinations as in Theorem 3.1 then*
(a) $D_6 = \{a_0; a_1, a_2, \ldots, a_m; b_1, b_2, \ldots, b_n; c_1, c_2, \ldots, c_r\}$ is graceful.
(b) $D_6 = \{a_0; a_1, \ldots, a_m; b_1, b_2, \ldots, b_n\}$ is graceful.

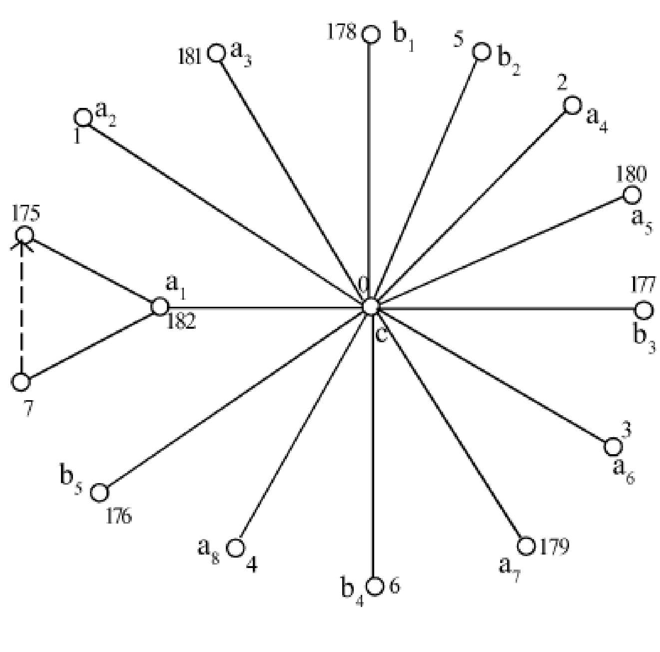
Proof. We prove the part (a) first. Let us designate the vertex b_n as the center of a star adjacent to a_0 with odd (respectively, even) degree if $n_e \equiv 1 \ mod\ 4$ (respectively, $n_e \equiv 0 \ mod\ 4$, $n_o \geq 1$). Let us define two integers k_1 and k_2 as $k_1 = \begin{cases} n_e - 1 \text{ if } n_e \equiv 1 \ mod\ 4 \\ n_e \quad \text{ if } n_e \equiv 0 \ mod\ 4 \text{ and } n_o \geq 1 \end{cases}$ and $k_2 = \begin{cases} n_o \quad \text{ if } n_e \equiv 1 \ mod\ 4 \\ n_o - 1 \text{ if } n_e \equiv 0 \ mod\ 4 \text{ and } n_o \geq 1 \end{cases}$

So we have $n = n_o + n_e = k_1 + k_2 + 1$. Form a diameter six tree, say G_6 by removing the vertices c_1, c_2, …, c_r, and b_n from D_6. Let $|E(G_6)| = q_1$. Give a graceful labeling to G_6 by following the steps 1 to 4 by setting $q - r = q_1$ and replacing n_e with k_1 and n_o with k_2 in the proof for Case -I of Theorem 3.1. Observe that in the graceful labeling of G_6, the vertex a_0 gets the label 0. Now attach the vertices $c_1, c_2, \ldots, c_r$, and b_n to a_0 and assign them the labels $q_1 + 1$, $q_1 + 2$, …, $q_1 + r$, and $q_1 + r + 1$, respectively. Obviously, the tree $G_6 \cup \{c_1, c_2, \ldots, c_r, b_n\}$ with the labelings mentioned above is graceful with a graceful labeling, say g. Then apply inverse transformation g_{q_1+r+1} to the above labeling of $G_6 \cup \{c_1, c_2, \ldots, c_r, b_n\}$. Now the vertex b_n gets the label 0. Let $deg(b_n) = p$. Finally, attach p pendant vertices to b_n and assign them the labels q_1+r+2, q_1+r+3, …, $q_1 + r + p + 1$, so as to get the tree D_6 with a graceful labeling. The proof of part (b) follows if we set $r = 0$. ■

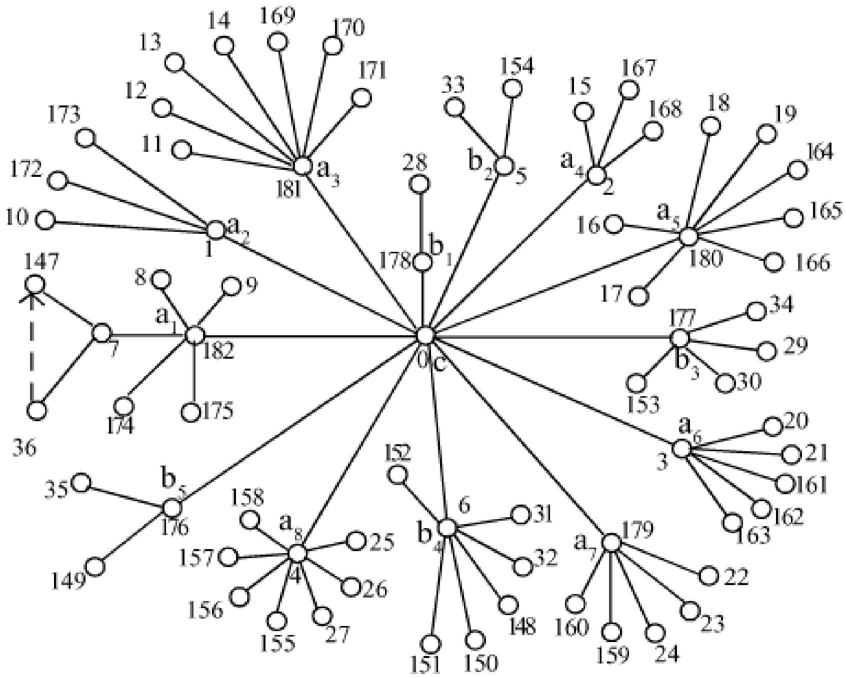
Example 3.3. Figure 8 (a) is a diameter six of the type in Theorem 3.3. Here $q = 192$, $m = 8$, and $n = 6, n_e = 5$, a_1 is attached to $(e, o, 0)$, each of a_2 is attached to $(0, o, 0)$, a_3 is attached to $(o, 0, e)$, a_4 is attached to $(o, e, 0)$, each of a_5 and a_6 is attached to (o, o, o), a_7 is attached to $(e, 0, o)$, and a_8 is attached to (e, e, o). We first form the graceful diameter six tree G_6 as in Figure (d) (without labeling) by removing all the pendant vertices and one star adjacent to c with odd degree. The transfer $T_1 : a_1 \rightarrow a_2 \rightarrow \ldots \rightarrow b_{n-1} \rightarrow z_1$ in Step 3 is the transfer $182 \rightarrow 1 \rightarrow 181 \rightarrow \ldots \rightarrow 176 \rightarrow 7$. $T_2 : z_1 \rightarrow z_2 \rightarrow \ldots \rightarrow z_s$ in Step 4 is the transfer $7 \rightarrow 175 \rightarrow 8 \rightarrow \ldots \rightarrow 27 \rightarrow 155$ We first form the graceful tree G as in Figure (b). Figure (c) represents the graceful tree G_1 obtained after Step 3. Figure (d) represents the graceful tree G_6 obtained after Step 4. Figure (e) represents the tree obtained from the graceful tree in (d) by attaching three pendant vertices to c and assigning them the labels 183, 184, and 185. Finally, the graceful tree in Figure (f) is obtained by applying inverse transformation to the graceful tree in Figure (e) (so that the label of the vertex b_6 becomes 0), and attaching eight vertices to the vertex b_6 (labelled 0) and assign them the labels 186, 187, 188, 189, 190, 191, 192, and 193. ■


a3
a4
b2
a2
a5
c
a1
b5
b3
b1
c1
c2
b4
a6
b6
a8
a7

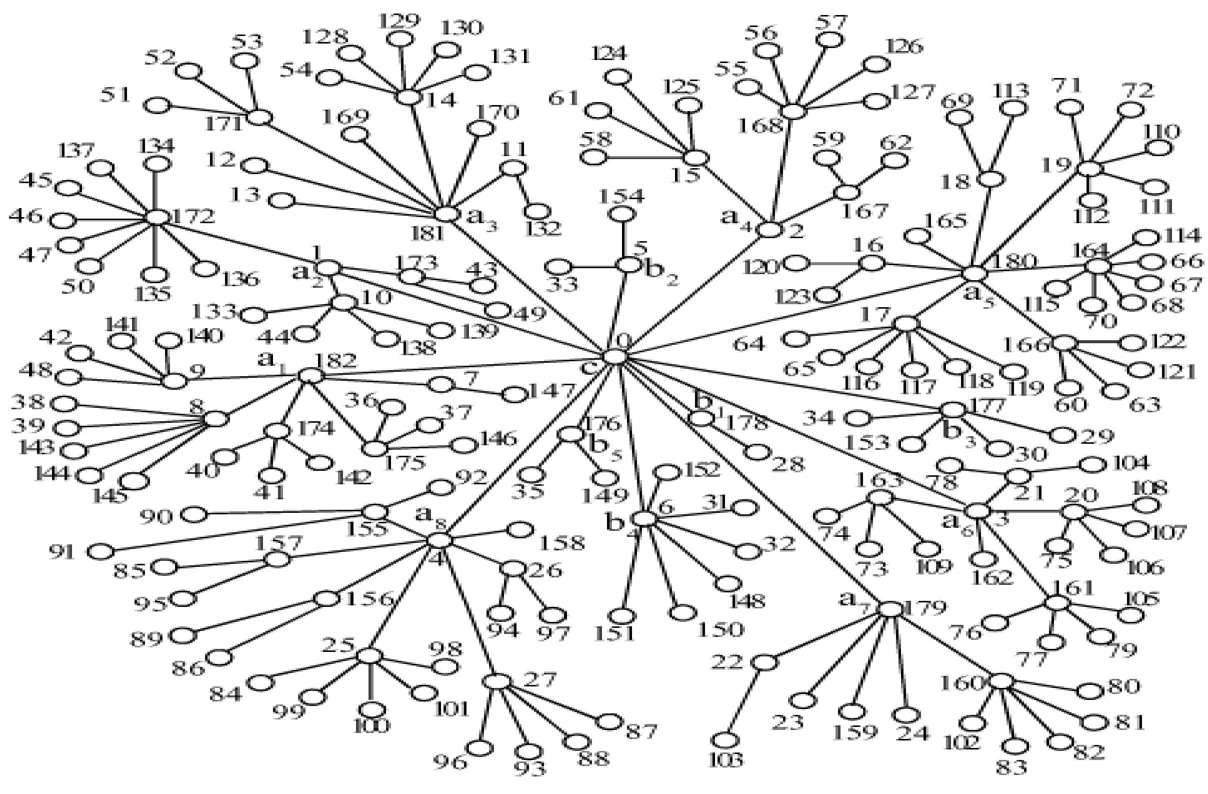
(a)


178 b1
181 a3
5 b2
2 a4
a2 1
180 a5
175
a1 182
0 c
177 b3
7
3 a6
b5 176
a8 4
b4 6
a7 179

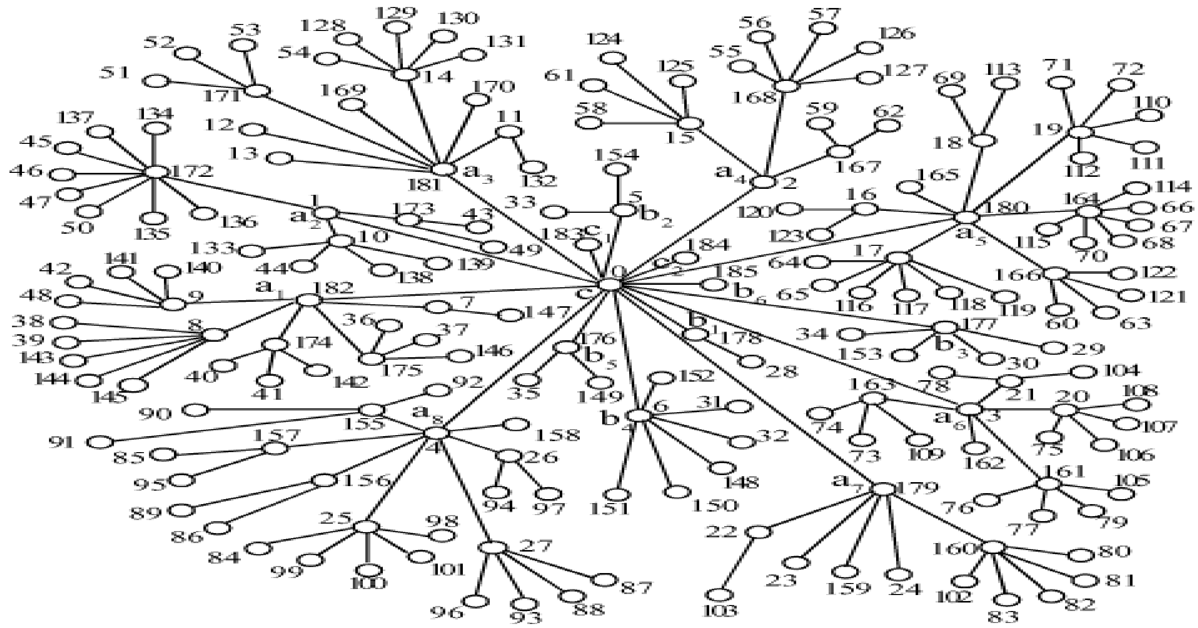
(b)


14
169
170
13
171
12
173
11
181
a_3
172
10
1
a_2
147
8
9
a_1
7
182
174
175
36
35
b_5
176
149
158
a_8
157
25
4
156
26
155
27
28
178
b_1
33
154
b_2
5
15
167
168
a_4
2
18
19
164
a_5
16
165
180
166
17
177
34
29
b_3
30
153
0
c
a_6
20
21
3
161
162
163
152
6
b_4
31
32
148
151
150
a_7
179
22
23
160
159
24

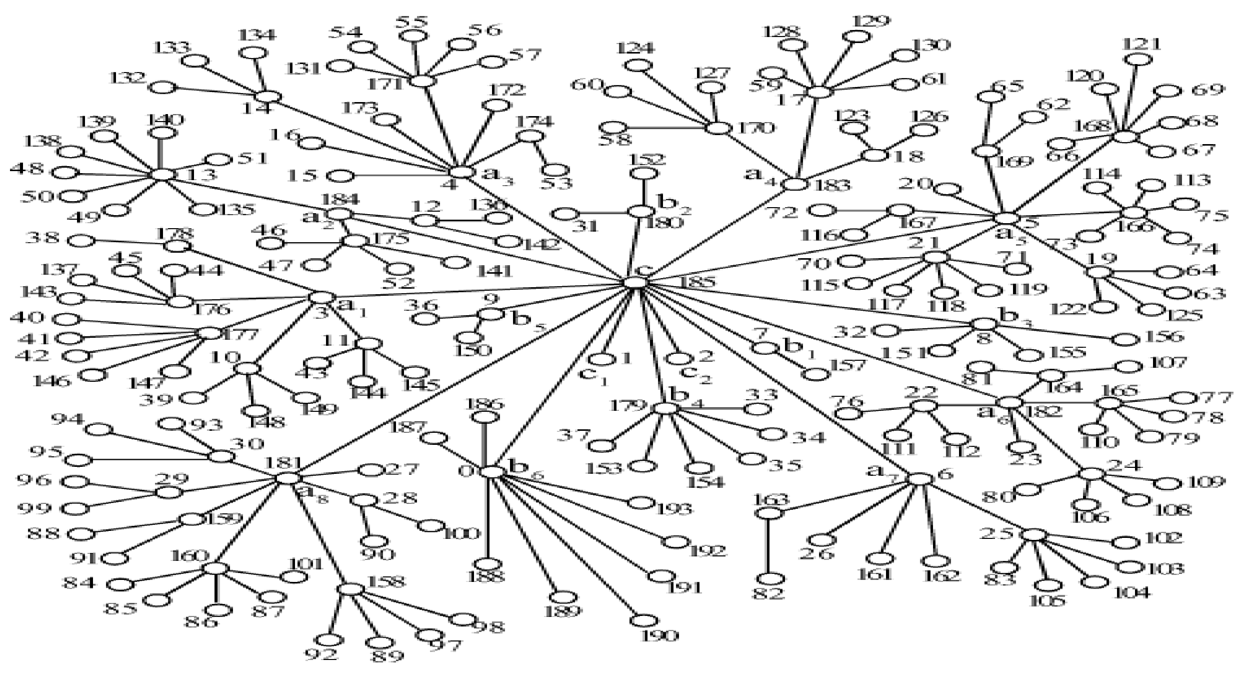
(c)


129
130
128
53
131
52
54
14
51
171
169
170
11
137
134
12
45
172
13
181
a_3
132
46
47
136
50
135
1
a_2
173
43
133
10
49
44
138
139
42
141
140
48
9
a_1
182
7
147
38
8
39
36
37
143
174
175
146
144
40
142
145
41
90
92
155
a_8
91
157
158
85
4
26
95
156
89
94
97
86
25
98
84
99
100
101
27
87
96
93
88
124
57
56
126
55
127
125
61
168
58
15
59
62
154
a_4
2
167
5
b_2
33
0
c
176
b_5
35
149
b_1
178
28
152
6
b_4
31
32
148
151
150
69
113
71
72
110
18
19
112
111
165
16
180
164
114
66
120
123
a_5
115
67
68
70
17
64
166
122
65
121
116
117
118
119
60
63
34
177
153
b_3
29
30
78
21
104
163
20
108
74
a_6
3
107
73
109
162
75
106
161
105
76
77
79
a_7
179
22
160
80
23
159
24
81
102
82
83
103

(d)



(e)



(f)

Figure 8. A diameter six tree of the type in Theorem 3.2 with a graceful labeling.

References

[1] V. Bhat-Nayak and U. Deshmukh, New families of graceful banana trees, *Proc. Indian Acad. Sci. Math. Sci.* **106** (1996), 201–216.

[2] W.C. Chen, H.I. Lu, and Y.N. Lu, Operations of Interlaced trees and graceful trees, *South East Asian Bulletin of Mathematics* **4** (1997), 337–348.

[3] M. Edwards and L. Howard, A survey of graceful trees, *Atlantic Electronic Journal of Mathematics* **1** (2006), 5–30.

[4] J.A. Gallian, A dynamic survey of graph labeling, *Electronic Journal of Combinatorics*, DS6, Sixteenth edition, December 20, 2013, url: `http://www.combinatorics.org/Surveys/`.

[5] P. Hrnciar and A. Haviar, All trees of diameter five are graceful, *Discrete Math.* **233** (2001), 133–150.

[6] J. Jeba Jesintha, *New Classes of Graceful Trees*, Ph. D. Thesis, Anna University, Chennai, India, 2005.

[7] J. Jeba Jesintha and G. Sethuraman, All arbitrarily mixed generalized banana trees are graceful, preprint.

[8] M. Murugan and G. Arumugam, Are banana trees graceful?, *Math. Ed. (Siwan)* **35** (2001), 18–20.

[9] D. Mishra and A.C. Panda, Some new transformations and their applications involving graceful tree labeling, *International Journal of Mathematical Sciences and Engineering Applications* **7** (1) (2013), 239–254.

[10] D. Mishra and P. Panigrahi, A new class of graceful lobsters obtained from diameter four trees, *Utilitas Mathematica* **80** (2009), 183 –209.

[11] D. Mishra and P. Panigrahi, Some graceful lobsters with all three types of branches incident on the vertices of the central path, *Computers and Mathematics with Applications* **56** (2008), 1382–394.

[12] E. Robeva, *An Extensive Survey of Graceful Trees*, Undergraduate Honours Thesis, Standford University, USA, June, 2011.

[13] A. Rosa, On certain valuations of the vertices of a graph, Theory of Graphs (Internat. Symposium, Rome, July 1966), Gordon and Breach, N.Y. and Dunod, Paris (1967), 349–355.

[14] G. Sethuraman and J. Teba Jesintha, All extended banana trees are graceful, *Proc. Internat. Conf. Math. Comput. Sci.* **1** (2009), 4–8.

[15] G. Sethuraman and J. Jeba Jesintha, All banana trees are graceful, *Advances Appl. Disc. Math.* **4** (2009), 53–64.

[16] V. Vilfred and T. Nicholas, Banana trees and unions of stars are integral sum graphs, *Ars Combin.* **102** (2011), 79–85